ADAPTIVE CONTROL OF TIME-VARYING SYSTEMS WITH APPLICATIONS TO MISSILE AUTOPILOTS

By

CHANHO SONG

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1989

To My Parents

ACKNOWLEDGEMENTS

I would like to thank my advisor and committee chairman, Professor T. E. Bullock, for his guidance, encouragement and support. Especially, I am grateful to him for many helpful insights and discussions about estimation algorithms. My deep gratitude is given to Professor E. W. Kamen, who introduced me to the basic idea of this dissertation. I express also my gratitude to Professor S. A. Svoronos in the chemical engineering for valuable discussions and suggestions. I would like also to thank Professors J. Hammer, H. A. Latchman and U. H. Kurzweg for being members of my graduate committee.

Of course, this work could not been completed without the help of my wife. My special thanks are given to her. Finally, I express my gratitute to the US Air Force, AFATL, Eglin AFB, Florida, for supporting this work under Contract No. F08635-86-K-0265.

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Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

ADAPTIVE CONTROL OF LINEAR TIME-VARYING SYSTEMS
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By

CHANHO SONG

December 1989

Chairman: Professor T. E. Bullock
Major Department: Electrical Engineering

In this dissertation an indirect adaptive control algorithm is presented, which can be applied to linear time-invariant systems as well as to linear time-varying systems with rapidly varying parameters. This algorithm does not require persistent excitation. A key feature of this algorithm is the use of an assumed structure of parameter variations which enables it to cover a wide class of linear systems.

A modification of the least squares algorithm is used for the parameter estimation, which has a dead zone to handle robustness to the unmodelled plant uncertainties or external disturbances. A forgetting factor is introduced in the algorithm to aid in the tracking of time-varying parameters. The forgetting factor used is based on the constant information principle. The control law used is due to Kleinman. The advantage of this method is that it can be applied to linear time-varying plants with reachability index greater than the system order n as well as to the plants with reachability index equal to n.

With this control law and a state observer, all based on the parameter estimates, it is shown that the resulting closed-loop system is robustly globally stable. As a design example, the proposed algorithm is applied to the design of the normal acceleration controller of missiles with rapidly changing parameters. Excellent performance is demonstrated by the proposed algorithm.

CHAPTER ONE
INTRODUCTION

Although the main applications of adaptive control are in situations where plant characteristics or parameters are changing, the strongest theoretical results have been made for linear time-invariant plants. Most research in the adaptive control of time-varying systems assumes that the system is slowly time-varying [1, 2]. Only a few papers consider time-varying systems which are not necessarily slowly time-varying. First results in this direction were reported by Xianya and Evans [3]. They modelled the unknown parameter variations as linear curves. Later, Zheng [4] replaced linear curves by the truncated Taylor series. This motivated the concept of the *structured parameter variations* which play an important role in this dissertation. But because Xianya and Evans, and Zheng chose one-step-ahead controller structure, they only treated a class of stably invertible systems. Moreover, they did not consider the robustness problem to the unmodelled plant uncertainties or external disturbances.

More recently, Tsakalis and Ioannou [5] proposed an indirect adaptive pole placement scheme for continuous linear time-varying plants which may be fast time-varying. They used the same concept of structured parameter variations as this dissertation. However, it is not clear how to make the estimated model reachable [5]. If the estimated model is not reachable, the closed-loop stability is not guaranteed. On the other hand, the pole placement scheme [5] requires that the plant to be controlled should be reachable in n steps where n is the plant order. This assumption

will be weakened to the N-step reachability where N is a positive integer with $N \geq n$ in this dissertation.

There are two main difficulties that any indirect adaptive control scheme should overcome. One is the reachability condition imposed on the estimated model. In order to construct a control law for the estimated model, it should be reachable, or at least stabilizable if the plant is linear time-invariant. The other is the robustness problem which has been intensively investigated since the work of Rohrs et al. [6]. The closed-loop should be stable in the presence of unmodelled dynamics or external disturbances.

The attempts which have been made for the reachability condition are as follows. In earlier works of indirect adaptive control for linear time-invariant plants, Samson and his coworker [7, 8, 9] assumed that the estimated model is uniformly reachable as a generic property. But later, to ensure the reachability of the estimated model, a convex set of parameters whose element corresponds to a system satisfying this condition was introduced. A projection scheme then was used to keep the parameter estimates within this set [10, 11], or a correction term was added in the standard estimation algorithm to drive them to this set gradually [12, 13].

Another approach is to use persistent excitation. Persistent excitation eventually makes the estimates converge to a certain tuned parameter matrix which is assumed to satisfy the reachability condition. However, persistent excitation needs an external signal which is sometimes undesirable from a control point of view [14].

Another alternative is to use search techniques in the parameter space in parallel with estimation algorithm to provide proper substitutes (call it modified estimates) which ensures the reachability of the estimated model whenever the estimates violate

the reachability condition [15, 16]. Lozano-Leal and Goodwin [16] suggested an algorithm to find the modified estimates which converge for the time-invariant plant. But when applied to time-varying plants modified estimates may not converge even though the estimates converge.

For the robustness of indirect or direct adaptive control schemes, several attempts have been made. The first one is to produce persistent excitation which will give the tuned model exponential stability. This, in turn, will make the plant stable in the presence of bounded disturbances. However, guaranteeing the persistent excitation in the presence of unmodelled plant uncertainties still remains unresolved [17].

The second one is σ-modification proposed by Ioannou and Kokotovic [18] which is a direct adaptive control law with an extra term $-\sigma\theta$, $\sigma > 0$. It was shown that robustness to the plant uncertainties can be obtained using σ-modification for time-invariant plants [19].

The third one is signal normalization by which unmodelled plant uncertainties can be viewed as bounded disturbances. Middleton and Goodwin showed that robustness to the plant uncertainties can be obtained by properly normalizing the signals which are used in the estimation algorithm, but it does not guarantee the boundedness of the parameter estimates [14].

The fourth one is to use a dead zone. When the prediction error is less than a certain threshold the estimates may be in the "wrong direction" due to the poor information spoiled with disturbances or unmodelled plant uncertainties. This can even make the estimates diverge. Thus, by using a proper dead zone estimation can be always in the "right direction," and robustness to the plant uncertainties can be

obtained. Sometimes, some of these ideas are incorporated. For example, a dead zone with signal normalization was used by Kreisselmeir [17].

In this dissertation, we consider an indirect adaptive control scheme for discrete linear time-varying plants which are reachable in N steps where N is greater than or equal to the plant order, and not necessarily slowly time-varying nor stably invertible. It is assumed that the plant is described by a linear vector difference equation and the time variation in coefficients is modelled as a linear combination of known bounded functions with unknown coefficients which are constant or allowed to be slowly time-varying. The unknown coefficients are then estimated by an estimation scheme based on the least squares algorithm, but properly modified as follows:

- A projection method is used in order to ensure the reachability of the estimated model [1, 10, 11].
- For the robustness to the unmodelled plant uncertainties and external disturbances, all the signals used in the estmation algorithm are normalized with respect to the norm of the regression vector and a dead zone is used. Signal normalization will make the size of dead zone bounded.
- A forgetting factor is used. It will make the estimation algorithm track slowly time-varying parameters and thus the unknown coefficients to be estimated will be allowed to be slowly time-varying. In consequence, we can expect that the unmodelled plant uncertainties will be smaller than otherwise. This is preferable because we are allowed to take the smaller size of dead zone in the estimation algorithm.

In fact, these modifications are not new. They appear frequently in the literature, separately or partially together with the least squares algorithm. However, we combine all of them to make a form which is useful in practice, and give the complete convergence analysis.

As a control law, a generalized Kleinman's method [20] is taken. It allows us to have a weaker assumption about reachability, i.e., the assumption that the plant is reachable in N steps with $N \geq n$, where n is the plant order. With this control law and a state observer, all based on the estimated parameter values, it is shown that the resulting closed-loop system is robustly globally stable in the sense that all signals are bounded if the reference input is bounded, in the presence of plant uncertainties and external disturbances.

As a design example, the proposed adaptive control algorithm is applied to the design of the normal acceleration controller of a typical missile whose dynamic pressure (thus plant parameters) changes very rapidly. Excellent performance is demonstrated by the proposed algorithm.

This dissertation is organized as follows. In Chapter Two, we review some basic adaptive control schemes and briefly discuss two important problems of adaptive control, robustness and time-varying plants with rapidly varying parameters, which lead to main subjects of this dissertation.

In Chapter Three, a mathemetical model of the plant with assumptions and the control objective of the proposed algorithm are presented. In Chapter Four, an estimation algorithm for the plant parameters, which is based on the least squares algorithm for multi-output systems but includes a forgetting factor, dead zone and

a projection operator, will be introduced and some important properties of this algorithm will be shown.

Chapter Five deals with a state feedback control law for the estimated model which is based on Kleinman's law and a state observer for the plant, all based on the estimated parameters. With the parameter estimation algorithm described in Chapter Four, it is shown that we can construct a bounded sequence of feedback gains which makes the estimated model exponentially stable. In Chapter Six, it is shown that the control law constructed in Chapter Five with a state observer, all based on the parameter estimates, the closed-loop system is globally stable and robust to the unmodelled plant uncertainties and the external disturbances.

In Chapter Seven, the proposed algorithm is implemented in the normal acceleration controller for a typical air-to-air missile and the performance is demonstrated. Many practical problems are discussed. Finally, conclusions are given in Chapter Eight.

CHAPTER TWO
BACKGROUND

In this chapter, we first review some basic adaptive control schemes, especially Samson and Fuchs' work [7] which is the basis of this dissertation. After that, we briefly discuss two important problems of adaptive control, robustness and time-varying plants which are not slowly time-varying, which lead to main subjects of this dissertation. Since the aim of the review is to grasp the basic ideas of adaptive control algorithms, we only consider the algorithms for LTI (Linear Time-Invariant) systems which were reported before the robustness problem was raised. In fact, these algorithms bave been modified to get better robustness, but these modifications are not included in this review.

2.1 Review of Basic Adaptive Control Schemes

We begin with a simple scheme called *Gain Scheduling* which is very old but is still used, especially for handling parameter variations in the flight control systems. The concept is based on "frozen coefficients." A family of controllers is designed as a function of unknown constant plant parameters. Implementation requires direct measurement or estimation of these parameters and a subsequent adjustment of the controller. The resulting controller is actually time-varying because either the measured plant parameters or the estimated values are usually not constant.

In general, a control scheme is said *adaptive* if it consists of two parts, a parameter estimator which estimates plant parameters or control parameters and a

controller which is adjusted with the estimated values of plant parameters or control parameters. In consequence, there are two feedback loops in an adaptive control system. The inner loop is an ordinary feedback loop with a plant and a controller. The outer loop is an adaptation mechanism which adjusts the controller.

In gain scheduling, there is no feedback to compensate for an incorrect schedule. Therefore, according to the above definition, it is not included in the category of adaptive control algorithms.

Even though gain scheduling has been used for some time, it was not until the late 1970s that the stability problem was rigorously treated in the context of slowly time-varying systems [21]. The stability for time-varying systems which are not slowly time-varying is not guaranteed with gain scheduling.

Now, we turn to the modern adaptive control schemes. In general, an adaptive control scheme is called direct if control parameters are directly updated, and it is called indirect if plant parameters are estimated and using these parameter estimates control parameters are obtained. There are two basic problem formulations in modern adaptive control. The one is the MRAC (Model Reference Adaptive Control) and the other is the STC (Self-Tuning Control).

In the MRAC, a reference model is properly selected and the control parameters are adjusted such that the error between the model output and the plant output becomes small. This scheme requires that the plant to be controlled is stably invertible, because this approach is based on the cancellation of zeros of the plant. If control parameters are directly updated, it is called a direct MRAC, otherwise it is called an indirect MRAC. The research in this direction has been done mainly

for direct MRACs of continuous LTI systems. A typical example is an algorithm published by Narendra and Valavani [22].

The idea of STC originated from the minimum variance self-tuner which was developed to solve the stochastic regulation problem for unknown discrete time-invariant systems [23]. In minimum variance self-tuners, the control parameters are estimated by a recursive parameter estimator such that the variance of the output is minimized. However, thereafter many researchers have widely extended this concept, e.g., pole placement self-tuners [24], model reference self-tuners, LQG (Linear Quadratic Gaussian) self-tuners [25] and so on. The terminology STC is used as having the almost same meaning as adaptive control. Usually, the direct approach is called an implicit STC and the indirect approach is called an explicit STC. In the case of implicit STC, stability is guaranteed if the plant to be controlled is minimum-phase. In contrast to MRAS, the minimum-phase condition comes from a desire to make the estimation model linear in the parameters. For nonminimum-phase plants, the estimation model becomes nonlinear in parameters to be estimated, which makes the parameter estimation very difficult. To solve this problem, nonlinear estimation schemes have been suggested [26]. Elliot proposed a model for parameter estimation which is linear in parameters using the Bezout identity [27]. However, it has been reported by Johnson, Lawarence and Lyons [28] that when the plant order is underestimated Elliot's algorithm can fail to work.

Kreisselmeir's indirect adaptive regulator [29] is another direction of research. It consists of an adaptive state observer and an asymptotic feedback matrix synthesis. Samson's indirect approach [8] is similar to Kreisselmeir's in that it also uses an adaptive state observer, but the control law is constructed in a different way. In

Samson's approach, feedback gains are obtained by solving one step of a Riccati difference equation at each time. This approach is also used by Samson and Fuchs [7] and Ossman and Kamen [13]. Since Samson and Fuchs' work [7] is the basis of our adaptive control algorithm, we will treat more details in Section 2.5.

2.2 An Example of Direct MRAC

Consider the following SISO (Single Input Single Output) plant

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k) \tag{2.1}$$

where q^{-1} is the unit delay operator, $u(k)$ and $y(k)$ denote the input and output, respectively, and $A(q^{-1})$ and $B(q^{-1})$ are given by

$$A(q^{-1}) = 1 + a_1q^{-1} + a_2q^{-2} + \cdots + a_nq^{-n} \tag{2.2}$$

$$B(q^{-1}) = b_0 + b_1q^{-1} + b_2q^{-2} + \cdots + b_mq^{-m} \tag{2.3}$$

$$b_0 \neq 0\ ;\ d \geq 1\ ;\ m + d \leq n.$$

We need the following assumptions:

A1. An upper bound for n is known.

A2. d is known.

A3. The plant is minimum phase, i.e., $B(z^{-1})$ is Hurwitz.

A4. b_0 is known.

Remark 2.1. For simplicity, we impose A4 as [30]. However, Assumption A4 can be weakened if the sign of b_0 and an upper bound for the magnitude of b_0 is known. See, for example, Goodwin and Sin [11].

Let the reference model be given by

$$A_m(q^{-1})y_m(k) = q^{-d}B_m(q^{-1})r(k) \tag{2.4}$$

where $r(k)$ and $y_m(k)$ denote the reference input and output, respectively. We further assume that

$$deg(A_m(z)) = deg(A(z))$$

where $deg(A(z))$ denotes a degree of a polynomial $A(z)$.

Suppose that the control law is of the form

$$R(q^{-1})u(k) = T(q^{-1})r(k) - S(q^{-1})y(k) \tag{2.5}$$

where the coefficients of $R(q^{-1})$, $T(q^{-1})$ and $S(q^{-1})$ are to be determined so that the closed-loop has the same transfer function as the reference model (2.4). By equating the closed-loop transfer function to that of the reference model, we have

$$\frac{q^{-d}T(q^{-1})B(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-d}B(q^{-1})S(q^{-1})} = \frac{q^{-d}B_m(q^{-1})}{A_m(q^{-1})} \tag{2.6}$$

which gives a solution:

$$T(q^{-1}) = B_m(q^{-1}) \tag{2.7}$$

$$A(q^{-1})R_1(q^{-1}) + q^{-d}S(q^{-1}) = A_m(q^{-1}) \tag{2.8}$$

with

$$R(q^{-1}) = B(q^{-1})R_1(q^{-1}) \tag{2.9}$$

where

$$A_m(q^{-1}) = 1 + c_1q^{-1} + \cdots + c_nq^{-n}$$

$$S(q^{-1}) = s_0 + s_1q^{-1} + \cdots + s_{n-1}q^{-n+1}$$

$$R(q^{-1}) = r_0 + r_1q^{-1} + \cdots + r_{m+d-1}q^{-m-d+1}.$$

From (2.8) and (2.9), we see that $r_0 = b_0$. Multiplying $y(k)$ on both sides of (2.8) and using (2.1) gives

$$q^{-d}R(q^{-1})u(k) + q^{-d}S(q^{-1})y(k) = A_m(q^{-1})y(k). \tag{2.10}$$

Let $s_i(k-1), i = 0, ..., n-1$ and $r_i(k-1), i = 1, ..., m+d-1$ denote the estimated values of $s_i, i = 0, ..., n-1$ and $r_i, i = 1, ..., m+d-1$ at time $k-1$, respectively. Then the error equation

$$e(k) = A_m(q^{-1})y(k) - b_0u(k-d) - \phi'(k-1)\theta(k-1) \tag{2.11}$$

where

$$\begin{aligned} \phi'(k-1) &= [\, y(k-d) \cdots y(k-n-d+1) \\ &\qquad u(k-d-1) \cdots u(k-m-2d+1)\,] \end{aligned} \tag{2.12}$$

$$\theta'(k-1) = [\, s_0(k-1) \cdots s_{n-1}(k-1)\, r_1(k-1) \cdots r_{m+d-1}(k-1)\,]. \tag{2.13}$$

is used to calculate $\theta(k)$ recursively (see Section 2.4 for estimation algorithms). From the equation (2.5), the control input based on the current estimate $\theta(k)$ is given by

$$u(k) = \frac{1}{b_0}\left(- \sum_{i=1}^{m+d-1} r_i(k)u(k-i) + T(q^{-1})r(k) - \sum_{i=0}^{n-1} s_i(k)y(k-i) \right). \tag{2.14}$$

2.3 An Example of Indirect Pole Placement Adaptive Control

Consider the plant

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k) \tag{2.15}$$

where $u(k)$ and $y(k)$ denote the input and output, respectively, and $A(q^{-1})$ and $B(q^{-1})$ are polynomials in q^{-1} given by

$$A(q^{-1}) = 1 + a_1q^{-1} + a_2q^{-2} + \cdots + a_nq^{-n} \tag{2.16}$$

$$B(q^{-1}) = b_1q^{-1} + b_2q^{-2} + \cdots + b_mq^{-m}. \tag{2.17}$$

We need the following assumptions:

A1. $r = \max\{n, m\}$ is known.

A2. $A(q^{-1})$ and $B(q^{-1})$ are coprime.

Compare these assumptions with those of the previous example. From A2, we know that there exist polynomials $E(q^{-1})$ and $F(q^{-1})$ which satisfy

$$A(q^{-1})E(q^{-1}) + B(q^{-1})F(q^{-1}) = A^*(q^{-1}) \tag{2.18}$$

where $A^*(q^{-1})$ is a desired characteristic polynomial of order $2r-1$. The pole placement control law is then given by

$$E(q^{-1})u(k) = F(q^{-1})(r(k) - y(k)) \tag{2.19}$$

where $r(k)$ is the reference input. Let $a_i(k-1)$ and $b_i(k-1)$ be the estimates of a_i and b_i at time $k-1$, respectively. Then, in indirect approach, the error equation (called prediction error equation) is written by

$$e(k) = y(k) - \phi'(k-1)\theta(k-1) \tag{2.20}$$

where

$$\phi'(k-1) = [\, y(k-1) \ \cdots \ y(k-r) \ u(k-1) \ \cdots \ u(k-r) \,] \tag{2.21}$$

$$\theta'(k-1) = [\, a_1(k-1) \cdots a_r(k-1) \, b_1(k-1) \cdots b_r(k-1) \,]. \tag{2.22}$$

The equation (2.20) is used to calculate $\theta(k)$ (see the next section for estimation algorithms). Now, $A(q^{-1})$ and $B(q^{-1})$ with coefficients replaced by estimated parameters may not be coprime, which will make it difficult to calculate control inputs. To ensure the coprimeness of the estimated $A(q^{-1})$ and $B(q^{-1})$, several methods explained in Chapter One (for example a projection scheme) can be used.

2.4 Basic Estimation Algorithms

When $e(k)$ denotes the error equation (2.11) or (2.20), The following 2 algorithms are commonly used to calculate $\theta(k)$ in adaptive control algorithms [11, 31]:

Gradient Algorithm.

$$\theta(k) = \theta(k-1) + \frac{\phi(k-1)e(k)}{c + \phi'(k-1)\phi(k-1)}\ ;\ c > 0. \tag{2.23}$$

Least Squares Algorithm.

$$\theta(k) = \theta(k-1) + P(k)\phi(k-1)e(k) \tag{2.24}$$

$$P(k) = P(k-1) \quad - \quad \frac{P(k-1)\phi(k-1)\phi'(k-1)P(k-1)}{1 + \phi'(k-1)P(k-1)\phi(k-1)} \tag{2.25}$$

$$P(0) = P'(0) > 0.$$

2.5 Samson and Fuchs' Work

In this section, we briefly review Samson and Fuchs' Work [7] which is the basis of this dissertation. Consider a SISO LTI plant described by

$$y(k) = \sum_{i=1}^{p} a_i\, y(k-i) + \sum_{i=1}^{q} b_i\, u(k-i) \tag{2.26}$$

where $u(k)$, $y(k)$ denote the scalar input and output, respectively, p and q denote fixed positive integers, and a_i, b_i are constant coefficients. We need following assumptions:

A1. $n = \max\{p, q\}$ is known.

A2. The plant is stabilizable.

The equation (2.26) can be written as

$$y(k) = \phi'(k-1)\theta \tag{2.27}$$

where

$$\phi'(k-1) = [\, y(k-1) \ \cdots \ y(k-n) \ u(k-1) \ \cdots \ u(k-n) \,] \tag{2.28}$$

$$\theta' = [\, a_1 \ \cdots \ a_n \ b_1 \ \cdots \ b_n \,]. \tag{2.29}$$

Note that $a_i = 0$ if $i > p$ and $b_i = 0$ if $i > q$. Then, the prediction error equation is given by

$$e(k) = y(k) - \phi'(k-1)\theta(k-1) \tag{2.30}$$

where $\phi(k)$ is as in (2.28) and $\theta(k-1)$ is the parameter estimate at time $k-1$ described as

$$\theta'(k-1) = [\, a_1(k-1) \ \cdots \ a_n(k-1) \ b_1(k-1) \ \cdots \ b_n(k-1) \,]. \tag{2.31}$$

To obtain the parameter estimate $\theta(k)$, a class of estimation algorithms which satisfy the following properties are considered:

P1. $\quad |e(k)| \leq c_1(k)\|\phi(k-1)\| + c_2(k) \quad$ (2.32)

where $\{c_1(k)\}$ and $\{c_2(k)\}$ are uniformly bounded,

$c_1(k) \to 0, \ c_2(k) \to 0, \ \text{as } k \to \infty.$

P2. $\quad \|\theta(k)\| < M \ \ \forall k, \ \text{ for some } \ M. \quad$ (2.33)

P3. $\quad \theta(k)$ converges to a constant vector. $\quad$ (2.34)

The least squares algorithm given in Section 2.4 satisfies all of these properties for the plant model (2.27) (see, for details, [8]).

Let $\hat{x}(k)$ denote the estimated state vector of the plant. Then a state observer can be constructed as

$$\hat{x}(k+1) = F(\theta(k))\hat{x}(k) + G(\theta(k))u(k) + M(k)v(k) \tag{2.35}$$

where

$$v(k) = y(k) - H\hat{x}(k) \tag{2.36}$$

$$F(\theta(k)) = \begin{bmatrix} a_1(k) & 1 & 0 \cdots 0 \\ a_2(k) & 0 & 1 \cdots 0 \\ \vdots & & \\ a_{n-1}(k) & 0 & 0 \cdots 1 \\ a_n(k) & 0 & 0 \cdots 0 \end{bmatrix} \tag{2.37}$$

$$G(\theta(k)) = \begin{bmatrix} b_1(k) \\ b_2(k) \\ \vdots \\ b_{n-1}(k) \\ b_n(k) \end{bmatrix} \tag{2.38}$$

$$H = [\, 1 \; 0 \cdots 0 \,]. \tag{2.39}$$

$M(k)$ is selected such that all of the observer poles are located at the origin, i.e.,

$$M'(k) = [\, a_1(k) \, a_2(k) \; \cdots \; a_n(k) \,]. \tag{2.40}$$

With this setup, we have the following theorem from [7].

Theorem 2.1. With the parameter estimation algorithm which satisfies 3 properties described above, with any control law $L(k)$ which is bounded and makes the system

$$F(\theta(k)) - G(\theta(k))L(k)$$

exponentially stable, the feedback controller

$$u(k) = -L(k)\hat{x}(k)$$

makes the resulting closed-loop system globally stable.

2.6 Robustness to Unmodelled Dynamics

In the early 1980s, it was argued that existing adaptive control algorithms could not be used with confidence in practical designs where the plant contained unmodelled dynamics [6]. The problem is that external disturbances or unmodelled dynamics can cause the estimated parameters to grow without bound [6]. Since then, considerable research has been done to get better understanding about the robustness of adaptive control algorithms. The following simple example shows how the estimated parameters can diverge.

Consider the SISO plant model

$$y(k) = \phi'(k-1)\theta + \delta(k) \tag{2.41}$$

with bounded external disturbances $\delta(k)$. Using the prediction error $e(k)$ given in (2.30) and the least squares algorithm (2.24) we can get (for details, see [32])

$$\theta(k) = \theta + (\sum_{i=0}^{t-1} \phi(i)\phi'(i) + P^{-1}(0))^{-1}(\sum_{i=0}^{t-1} \phi(i)\delta(i+1) + P^{-1}(0)(\theta(0) - \theta)). \tag{2.42}$$

Let $\phi(i) = 1/(i+1)$, $P(0) = 1$ and $\delta(i) = 1$. Then since

$$\sum_{i=0}^{t-1} \phi(i)^2 + P^{-1}(0) \;\rightarrow\; \frac{\pi^2}{6} + 1$$

and

$$\sum_{i=0}^{t-1} \phi(i)\delta(i+1) \;\rightarrow\; \infty$$

$\theta(k)$ diverges (see the Riemann zeta function for the value of $\sum_{i=1}^{\infty} i^{-2}$).

2.7 Time-Varying Plants

It is well known that a linear time-varying system

$$x(k+1) = A(k)x(k)$$

may not be stable, even though all eigenvalues of $A(k)$ have magnitudes less than one. The following is an example.

Example 2.1. Define

$$odd(k) = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

and consider the plant

$$y(k+1) = A(k)y(k) + Bu(k) \tag{2.43}$$

where

$$A(k) = \begin{bmatrix} 0 & 2\,odd(k) \\ 2\,odd(k+1) & 0 \end{bmatrix}, \; B(k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

All eigenvalues of $A(k)$ are zero at any k, but this plant is unstable. It implies that this plant is not slowly time-varying (see [33] for details of slowly time-varying

systems) and any controller based on the frozen coefficients will fail to make this plant stable. Therefore, any adaptive controller based on the frozen coefficients will also fail to work for the plant with parameter variations like this example. The use of the structure of parameter variations is a possible solution for this problem, which is one of main features of the proposed adaptive control algorithm.

2.8 Choice of Indirect Approach over Direct Approach

As mentioned in Section 2.1, direct approaches require a strong assumption that the plant to be controlled is stably invertible, while this assumption is not necessary for indirect approaches. As far as we know, there are no theories which support the stability of direct adaptive control for time-varying plants which are not slowly time-varying in the presence of unmodelled plant uncertainties. We therefore take an indirect approach.

CHAPTER THREE
PLANT MODEL AND THE CONTROL OBJECTIVE

3.1 Plant Model and Assumptions

We consider a single input m-output discrete linear time-varying plant described by

$$y(k) = \sum_{i=1}^{p} A_i(k)y(k-i) + \sum_{i=1}^{s} B_i(k)u(k-i) + d(k) \tag{3.1}$$

where $u(k)$, $y(k)$ denote the scalar input and m-dimensional output, respectively, $d(k)$ denotes the external disturbance, p and s denote fixed positive integers, and $A_i(k)$, $B_i(k)$ are time-varying matrices which have proper dimensions.

Assumption 1: The external disturbance $d(k)$ is bounded, i.e.,

$$\sup\{\|d(k)\|\} \leq W_1 \text{ for some } W_1. \tag{3.2}$$

Entries of $A_i(k)$ and $B_i(k)$ are assumed to be linear combination of certain known functions, i.e., the structure of the plant parameter variations are known a priori. We will call this type of parameter variations *Structured Parameter Variations* as Tsakalis and Ioannou [5]. In practice, there may be errors in modelling $A_i(k)$ and $B_i(k)$ by structured parameter variations. Therefore, we can write $A_i(k)$ and $B_i(k)$ in the form

$$A_i(k) = \sum_{j=1}^{r}(A_{ij} + \Delta A_{ij}(k))f_j(k) \quad \text{for } i = 1, \ldots, p \tag{3.3}$$

$$B_i(k) = \sum_{j=1}^{r}(B_{ij} + \Delta B_{ij}(k))f_j(k) \quad \text{for } i = 1, \ldots, s \tag{3.4}$$

where A_{ij} and B_{ij} are constant matrices (in practice, they are allowed to be bounded slowly time-varying matrices if the estimation algorithm can track time-varying parameters), r is a fixed positive integer, $f_j(k)$ are known functions of k which are linearly independent and bounded, for $j = 1, ..., r$, and $\Delta A_{ij}(k)$ and $\Delta B_{ij}(k)$ are terms due to modelling errors. We will call $\Delta A_{ij}(k)$ and $\Delta B_{ij}(k)$ *Unmodelled Plant Uncertainties* or simply *Plant Uncertainties.*

Now we define an $(mp+s)r \times m$ parameter matrix θ, an $(mp+s)r \times 1$ regression vector $\phi(k)$ and an $(mp+s)r \times m$ plant uncertainty matrix $\delta(k)$ by

$$\theta' = [A_{11} \cdots A_{1r} \cdots A_{p1} \cdots A_{pr} B_{11} \cdots B_{sr}] \tag{3.5}$$

$$\begin{aligned} \phi'(k-1) = {} & [f_1(k)y'(k-1) \cdots f_r(k)y'(k-1) \cdots f_1(k)y'(k-p) \cdots \\ & f_r(k)y'(k-p)\ f_1(k)u(k-1) \cdots f_r(k)u(k-1) \cdots \\ & f_1(k)u(k-s) \cdots f_r(k)u(k-s)] \end{aligned} \tag{3.6}$$

$$\begin{aligned} \delta'(k) = {} & [\Delta A_{11}(k) \cdots \Delta A_{1r}(k) \cdots \Delta A_{p1}(k) \cdots \Delta A_{pr}(k) \\ & \Delta B_{11}(k) \cdots \Delta B_{sr}(k)] \end{aligned} \tag{3.7}$$

where " $'$ " denotes the transpose of a matrix. Then (3.1) can be written in a compact form

$$y(k) = (\theta + \delta(k))'\phi(k-1) + d(k) \tag{3.8}$$

In the presence of unmodelled plant uncertainties, there is no $\theta \in \mathcal{R}^{(mp+s)r \times m}$ which satisfies

$$y(k) = \theta' \phi(k-1) + d(k).$$

Thus, we need to define a *tuned parameter matrix* which will be regarded as a reference for the estimation instead of the true parameters. We define a tuned

parameter matrix θ^* by a constant matrix satisfying

$$\|\delta(k)\| \text{ due to } \theta^* \leq \|\delta(k)\| \text{ due to any } \theta \in \mathcal{R}^{(mp+s)r \times m}.$$

where $\|\cdot\|$ denotes the Euclidean norm. From now on, $\delta(k)$ due to θ^* will be denoted by $\delta^*(k)$. Note that it is not necessary for the definition of the tuned parameters to be unique. See [34] for a different definition.

Assumption 2: Positive integers p, s, r are assumed known. Moreover, $p \leq n$, where n is the plant order.

Assumption 3: The tuned parameter matrix θ^* satisfies

$$\sup\{\|\delta^*(k)\|\} \leq D_1 \text{ for some } D_1 > 0. \tag{3.9}$$

To construct a state feedback control law, we need a state model of (3.1). We will take an observable realization for this purpose. We first consider the case that $p \geq s$. When we define n as

$$n = mp \tag{3.10}$$

an observable realization [35, 36] of (3.1) is given by

$$x(k+1) = F(k)x(k) + G(k)u(k) + \xi(k+1) \tag{3.11}$$

$$y(k) = Hx(k) \tag{3.12}$$

where an $n \times n$ matrix $F(k)$, an $n \times 1$ matrix $G(k)$ and an $m \times n$ matrix H are in the form

$$F(k) = \begin{bmatrix} A_1(k+1) & I & 0\cdots0 \\ A_2(k+2) & 0 & I\cdots0 \\ \vdots & & \\ A_{p-1}(k+p-1) & 0 & 0\cdots I \\ A_p(k+p) & 0 & 0\cdots0 \end{bmatrix}, G(k) = \begin{bmatrix} B_1(k+1) \\ B_2(k+2) \\ \vdots \\ B_{p-1}(k+p-1) \\ B_p(k+p) \end{bmatrix}$$

$$H = [\, I\, 0 \cdots 0\,]$$

and

$$\xi(k) = H'\, d(k). \tag{3.13}$$

Note that if $i > s$ $B_i(k+i) = 0$.

Now consider the case that $p < s$. If we define n as

$$n = ms$$

and take an observable realization, the resulting state model will not satisfy the reachability condition because the order of a reachable realization should not be greater than mp. This leads to the following lemma which gives a realization with the order mp. For convenience of notation, the summation $\sum_{i=t_0}^{t_1}$ will be taken as zero if $t_1 < t_0$.

Lemma 3.1. Suppose that for $q < p$,

$$y(k) = \sum_{i=1}^{p} A_i(k)y(k-i) + \sum_{i=1}^{p+q} B_i(k)u(k-i) + d(k) \tag{3.14}$$

and $A_p(k)$ is invertible for any k. Then there is a realization of (3.14) described by

$$x(k+1) = F(k)x(k) + G(k)u(k) + \xi(k+1) \tag{3.15}$$

$$y(k) = Hx(k) + \sum_{i=1}^{q} D_i(k)u(k-i) \tag{3.16}$$

where an $n \times n$ matrix $F(k)$, an $n \times 1$ matrix $G(k)$ and an $m \times n$ matrix H are in the form

$$F(k) = \begin{bmatrix} A_1(k+1) & I & 0 \cdots 0 \\ A_2(k+2) & 0 & I \cdots 0 \\ \vdots & & \\ A_{p-1}(k+p-1) & 0 & 0 \cdots I \\ A_p(k+p) & 0 & 0 \cdots 0 \end{bmatrix}, G(k) = \begin{bmatrix} G_1(k+1) \\ G_2(k+2) \\ \vdots \\ G_{p-1}(k+p-1) \\ G_p(k+p) \end{bmatrix}$$

$$H = [\, I \, 0 \cdots 0 \,]$$

and

$$\begin{aligned} \xi(k) &= H' \, d(k) \\ D_i(k-p) &= -A_p^{-1}(k)(B_{p+i}(k) + \sum_{j=1}^{q-i} A_{p-j}(k) D_{i+j}(k-p+j)) \\ &\quad \text{for } i = 1, \ldots, q \qquad (3.17) \\ G_i(k) &= B_i(k) - D_i(k) + \sum_{j=1}^{i-1} A_j(k) D_{i-j}(k-j) \\ &\quad \text{for } i = 1, \cdots, p, \quad D_i(k) = 0 \text{ if } i > q. \qquad (3.18) \end{aligned}$$

Proof(Outline). Using (3.15) and (3.16), express $y(k)$ in terms of $y(k-i)$, $i = 1, ..., p$. Equating this result to (3.14) gives (3.17) and (3.18). ■

Remark 3.1. In Lemma 3.1, the constraint that $q < p$ was used just because it was easy to handle. The case where $q \geq p$ can be treated in a similar way.

Using this lemma, we can find $F(k)$, $G(k)$ and $D_i(k)$, $i = 1, ..., q$ from $A_i(k)$, $i = 1, ..., p$ and $B_i(k)$, $i = 1, ..., p+q$. First, we find $D_q(k-p)$ from (3.17), i.e.,

$$D_q(k-p) = -A_p^{-1}(k) B_{p+q}(k).$$

Since we know the structure of the parameter variations, $D_q(k-p+1)$ follows from $D_q(k-p)$. Then, using $D_q(k-p+1)$, we obtain $D_{q-1}(k-p)$ from (3.17). Similarly, $D_i(k-p), i = 1, ..., q-2$ can be obtained. Now, the simple case where $m = 1$ is given in the following corollary. This corollary is not obtained directly from Lemma 3.1, because $q \not< p$, but can be easily proved in a similar way to Lemma 3.1.

Corollary. Suppose that

$$y(k) = A(k)y(k-1) + \sum_{i=1}^{q+1} B_i(k)u(k-i) + d(k) \tag{3.19}$$

and $A(k)$ is invertible for any k. Then there is a transformation

$$x(k) = y(k) - \sum_{i=1}^{q} D_i(k)u(k-i) \tag{3.20}$$

such that

$$x(k+1) = A(k+1)x(k) + B(k+1)u(k) + \xi(k+1) \tag{3.21}$$

$$D_i(k) = -A^{-1}(k+1)(B_{i+1}(k+1) - D_{i+1}(k+1))$$

$$\text{for } i = 1, ..., q, \; D_i(k+1) = 0 \text{ if } i > q \tag{3.22}$$

$$B(k) = B_1(k) - D_1(k) \tag{3.23}$$

$$\xi(k) = d(k).$$

From Lemma 3.1, we see that the realization (3.15)-(3.16) for $p < s$ has the same form as (3.11)-(3.12) for $p \geq s$ except some extra terms in the output equation.

Let a pair $(F(k), G(k))$ be a system described as

$$x(k+1) = F(k)x(k) + G(k)u(k) \tag{3.24}$$

and let $\Phi(k,i)$ denote the transition matrix of (3.24), i.e.,

$$\Phi(k,i) = F(k-1)F(k-2)...F(i) \text{ for } k > i; \; \Phi(i,i) = I.$$

For any positive integer N, let $Y_N(k)$ denote the N-step reachability grammian defined by

$$Y_N(k) = \sum_{i=k}^{k+N-1} \Phi(k+N, i+1)G(i)G'(i)\Phi'(k+N, i+1). \tag{3.25}$$

Then the system $(F(k), G(k))$ is called uniformly reachable in N steps if

$$Y_N(k) \geq r_1 I \text{ for some } r_1 > 0 \ \ \forall k \in \mathcal{Z}^+. \tag{3.26}$$

where $\mathcal{Z}^+$ denotes a set of nonnegative integers.

Remark 3.2. The definition of uniform reachability is due to [37].

Remark 3.3. If every element of $F(k), G(k)$ is bounded, $\mathrm{tr}(Y_N(k))$ is also bounded, where "tr" denotes the trace operation. Thus (3.26) holds if and only if there is $r_2 > 0$ such that $|\det(Y_N(k))| \geq r_2 \ \forall k \in \mathcal{Z}^+$.

Now we require the following.

Assumption 4: The observable realization of (3.1) described by (3.11)-(3.12) or (3.15)-(3.16) is uniformly reachable in N steps with $N \geq n$.

Remark 3.4. The observable realization is always uniformly observable in n steps. We, therefore, have a realization which is both uniformly reachable (in N steps) and uniformly observable (in n steps).

The assumption 4 seems too restrictive. In fact, an observable realization may not be reachable at all for certain systems. Nevertheless, we can reasonably treat more than single output systems ($m = 1$) and all-state measurable systems ($p = 1$) with this assumption. In order to treat more general cases, we can invoke the realization theories of linear time-varying systems, e.g., [38]. However, algorithms are generally too complicated to be used on-line, so we will not treat them in this dissertation.

In most cases, we have prior knowledge about the unknown parameter matrix θ^*. Thus, we assume that the ij-th element of θ^* denoted by θ_{ij} belongs to the known interval $[\ \theta_{ij}^{min}\ \ \theta_{ij}^{max}\ ]$ for all i,j [12, 13]. When we define the set Ω of all possible values of the unknown parameter matrix θ^* by

$$\Omega = \{\,\theta \in \mathcal{R}^{(n+s)r\times m} : \theta_{ij} \in [\,\theta_{ij}^{min}\ \theta_{ij}^{max}\,]\} \tag{3.27}$$

this can be written as the following.

Assumption 5: The tuned parameter matrix $\theta^* \in \Omega$.

Note that the set Ω is compact, thus closed and bounded.

Now consider

$$y(k) = \theta'\phi(k-1) \ \text{ for any } \ \theta \in \Omega. \tag{3.28}$$

To ensure the reachability of the estimated model, we make the following assumption.

Assumption 6: There is $\varepsilon > 0$ such that for any $\theta \in \Omega$, any $k \in \mathcal{Z}^+$ an observable realization of (3.28) satisfies

$$|\det(Y_N(\theta,k))| \geq \varepsilon \tag{3.29}$$

where $Y_N(\theta,k)$ is its reachability grammian.

When $p < s$, as can be expected in Lemma 3.1, we make the following additional assumption.

Assumption 7: If $p < s$, then there is $\varepsilon_1 > 0$ such that

$$|\det(A_p(\theta,k))| \geq \varepsilon_1 \ \ \forall \theta \in \Omega, \ \ \forall k \in \mathcal{Z}^+$$

where $A_p(\theta,k) = \sum_{j=1}^{r} A_{pj}\, f_j(k)$ and A_{pj}, $j = 1,\ldots,r$ are submatrices of θ' for any $\theta \in \Omega$.

3.2 Control Objective

Now the control problem is to find a proper adaptive control algorithm which makes the closed-loop system robustly globally stable. Robust stability means that there exists $D^* > 0$ such that for any bounded external disturbances $d(k)$ and for all possible plant uncertainties $\delta^*(k)$ which satisfy

$$\sup\{\|\delta^*(k)\|\} \leq D^*$$

the closed-loop system is stable [17].

CHAPTER FOUR
PARAMETER ESTIMATION

In this chapter, we introduce an estmation algorithm for θ^* and show some properties which will be used in proving the closed-loop stability. This estimation scheme is based on the least squares algorithm for multi-output systems (see pp 94-98, in Goodwin and Sin [10]) but includes a forgetting factor, a dead zone and a projection operator. In addition, all signals normalized by the norm of the regression vector will be used.

The motivation for using a forgetting factor is to allow the estimation algorithm to track slowly varying parameters. Conceptually, this implies that θ^* is allowed to be time-varying. Let $\theta^*(k)$ denote such a time-varying tuned parameter to distinguish it from the constant tuned parameter θ^*. Then sup $\|\delta^*(k)\|$ due to $\theta^*(k)$ will be smaller than sup $\|\delta^*(k)\|$ due to θ^*. In practice, this will allow us to have a smaller dead zone in the estimation algorithm. For certain plant models, it might be possible to track $(\theta^* + \delta^*(k))$ itself with small errors without a dead zone. However, in the ensuing analysis, we will treat only the constant tuned parameter θ^* because of technical difficulties.

To prevent the estimates from being updated in the wrong direction due to the disturbances or plant uncertainties, we use a dead zone. The size of dead zone is determined such that a Lyapunov function of the estimation error is monotonically decreasing. When the parameter estimates are outside of Ω, we project them onto the boundary of Ω to ensure the reachability condition (Assumption 6 in Chapter

Three) of the estimated model. A projection operator $\mathcal{P}$ will be used for this purpose.

The sequence of the algorithm is as follows. First, we normalize signals. Next, we calculate a forgetting factor $\lambda(k)$ and determine the value of a dead zone function represented by $\alpha(k)$ whose value is 0 or 1. After that, we update the covariance matrix $P(k)$ and the parameter estimate $\theta(k)$. Finally, if the updated parameter estimate is not in Ω, we project it onto Ω.

In Section 4.1, we show the formulas for $P(k)$ and $\theta(k)$. Next, we treat a projecton operator in Section 4.2, a dead zone in Section 4.3 and a forgetting factor $\lambda(k)$ in Section 4.4. Finally, we show some properties of this algorithm in Section 4.5.

4.1 Estimation Algorithm

First, we define the prediction error $e(k)$ by

$$e(k) = y(k) - \theta'(k-1)\phi(k-1) \tag{4.1}$$

where $\theta(k-1)$ is the estimate of the tuned parameter matrix θ^* at time $(k-1)$. Note that $e(k)$ is not a scalar but a vector since $y(k)$ is a vector. If some parameters are known a priori, we can replace (4.1) by

$$e(k) = y(k) - \psi'(k-1)\phi_a(k-1) - \theta'(k-1)\phi_b(k-1)$$

as [13], where $\psi(k)$ consists of the known parameters and $\phi_a(k)$, $\phi_b(k)$ are subvectors of $\phi(k)$.

We define normalized variables for $\phi(k)$, $y(k)$ and $e(k)$ as

$$\begin{aligned} w(k) &= \phi(k)/n(k) \\ \bar{y}(k) &= y(k)/n(k-1) \\ \bar{e}(k) &= e(k)/n(k-1) \end{aligned} \tag{4.2}$$

where

$$n(k) = \max\{1, \|\phi(k)\|\}. \tag{4.3}$$

Now the parameter estimation algorithm is given by

$$\theta(k) = \begin{cases} \mathcal{P}\{\theta_o(k)\} & \text{if } \theta_o(k) \notin \Omega \\ \theta_o(k) & \text{otherwise} \end{cases} \tag{4.4}$$

where

$$\theta_o(k) = \theta(k-1) + \alpha(k)P(k)w(k-1)\bar{e}'(k) \tag{4.5}$$

$$P(k) = \frac{1}{\lambda(k)}\{P(k-1) - \frac{\alpha(k)P(k-1)w(k-1)w'(k-1)P(k-1)}{\lambda(k)+\mu(k-1)}\} \tag{4.6}$$

$$P(0) = P'(0) > 0$$

$$\mu(k-1) = w'(k-1)P(k-1)w(k-1) \tag{4.7}$$

and $\lambda(k)$ is a forgetting factor with $0 < \lambda(k) \leq 1$, $\alpha(k)$ is a function which has a value 0 or 1 depending on the dead zone and $\mathcal{P}$ is a projection operator.

4.2 Projection Operator $\mathcal{P}$

In this section, we develop the projection operator $\mathcal{P}$ used in the parameter estimation algorithm (4.4). Suppose that $\theta_o(k)$ as in (4.5) is outside of Ω, so we want to get a substitute $\theta(k)$ for $\theta_o(k)$ such that $\theta(k)$ is included in Ω while retaining some important properties of the parameter estimation algorithm, for example, properties P1 - P3 in Section 2.5 in the absence of the plant uncertainties and external disturbances. When we choose a Lyapunov function $v(k)$ as

$$v(k) = \text{ tr } \{\tilde{\theta}'(k)P^{-1}(k)\tilde{\theta}(k)\} \tag{4.8}$$

with the estimation error $\tilde{\theta}(k)$ defined by

$$\tilde{\theta}(k) = \theta(k) - \theta^* \tag{4.9}$$

these properties are all retained if

$$v(k) \leq v_o(k)$$

where

$$v_o(k) = \text{ tr } \{\tilde{\theta_o}'(k)P^{-1}(k)\tilde{\theta}_o(k)\} \tag{4.10}$$

with

$$\tilde{\theta}_o(k) = \theta_o(k) - \theta^*. \tag{4.11}$$

This problem leads to the following projection method [10, 11]. We first transform the basis for the parameter space by the linear transformation $P^{-1/2}(k)$. Let $\rho_o(k)$ and $\bar{\Omega}(k)$ denote the images of $\theta_o(k)$ and Ω under $P^{-1/2}(k)$. Then, we can find $\theta(k)$ by orthogonally projecting each column vector of $\rho_o(k)$ onto the boundary of $\bar{\Omega}$ in the transformed space. We will describe this projection algorithm by using a projection operator $\mathcal{P}$ as

$$\mathcal{P}(\theta_o(k)) = \theta(k). \tag{4.12}$$

Now, it follows from the definition of $\mathcal{P}$ that for $\theta_o(k) \notin \Omega$

$$v_o(k) - v(k) \geq \text{tr}\{(\theta_o(k) - \theta(k))'P^{-1}(k)(\theta_o(k) - \theta(k))\}. \tag{4.13}$$

This problem can be reformulated as a problem of quadratic programming, i.e., to find a matrix $\theta(k)$ minimizing the performance index J_k given by

$$J_k = \text{tr}\{(\theta(k) - \theta_o(k))'P^{-1}(k)(\theta(k) - \theta_o(k))\} \tag{4.14}$$

such that $\theta_{ij} \in [\ \theta_{ij}^{min} \quad \theta_{ij}^{max}\ ]$ where θ_{ij} is the ij-th element of $\theta(k)$. We can obtain the solution by solving a finite number of equality constraint problems. A disadvantage of this method is that it may take too much time when the dimension is large. For example, if $\theta(k)$ and $\theta_o(k)$ are n-dimensional vectors and all the components of $\theta_o(k)$ are out of bound, then at worst the number of equality constraint problems we have to solve is

$$ {}_nC_1 + {}_nC_2 + \cdots + {}_nC_{n-2} + {}_nC_{n-1} $$

where ${}_pC_q$ is the q-th binomial coefficient for a p-th order polynomial. An alternative is to use an iterative method of quadratic programming. The choice will be problem dependent.

4.3 Dead Zone

In this section, we develop the dead zone function $\alpha(k)$ used in the covariance update equation (4.6). In order to prevent the parameter estimates from being updated in the wrong direction due to the external disturbances or plant uncertainties, we use a dead zone. The size of dead zone is selected such that the Lyapunov function defined as (4.8) is always monotonically decreasing in spite of external disturbances and plant uncertainties.

$\alpha(k)$ is defined by

$$ \alpha(k) = \begin{cases} 1 & \text{if } \Delta(k) \le \|\bar{e}(k)\| \\ 0 & \text{otherwise} \end{cases} \quad . \tag{4.15} $$

We select the size of dead zone $\Delta(k)$ as follows. Let W and and D be design parameters which satisfy

$$ W_1 \le W \quad \text{and} \quad D_1 \le D $$

where W_1 and D_1 are upper bounds in the norm of external disturbances and plant uncertainties, respectively, as in Assumption 1 and 3 of Chapter Three. With these design parameters, for any sequence of forgetting factors $\{\lambda(k)\}$ with $0 < \lambda(k) \leq 1$, we define $\Delta(k)$ by

$$\Delta(k) = \sqrt{1 + \frac{\mu(k-1)}{\lambda(k)}(\|w(k-1)\|D + W/n(k-1))} \tag{4.16}$$

where $w(k-1)$, $n(k-1)$ and $\mu(k-1)$ are as in (4.2), (4.3) and (4.7), respectively. From (4.5), if $\alpha(k)$ is equal to zero, $\theta_o(k)$ is not updated in spite of a non-zero prediction error $e(k)$ (or $\bar{e}(k)$). Thus $\alpha(k)$ can be regarded as a form of dead zone function.

As in Section 4.2, we define

$$\tilde{\theta}(k) = \theta(k) - \theta^* \tag{4.17}$$

$$\tilde{\theta}_o(k) = \theta_o(k) - \theta^*. \tag{4.18}$$

Then, the following lemma shows that a dead zone function defined above makes the Lyapunov function (4.8) monotonically decreasing.

Lemma 4.1. Consider the parameter estimation algorithm (4.4) - (4.7) applied to the plant (3.8) with the function $\alpha(k)$ defined in (4.15) where $D \geq D_1$ and $W \geq W_1$, and with a projection operator $\mathcal{P}$. For any sequence $\{\lambda(k)\}$ such that $0 < \lambda(k) \leq 1$, the function

$$v(k) = \mathrm{tr}\{\tilde{\theta}'(k)P^{-1}(k)\tilde{\theta}(k)\} \tag{4.19}$$

is nonnegative, monotonically decreasing and therefore converges.

Proof. Clearly, $v(k) \geq 0$, because $P(k) > 0$. Since $\alpha(k)$ has a value 0 or 1,

$$\alpha^2(k) = \alpha(k). \tag{4.20}$$

Postmultiplying (4.6) by $w(k-1)\alpha(k)$ and using (4.20), we get

$$\alpha(k)P(k)w(k-1) = \frac{\alpha(k)P(k-1)w(k-1)}{\lambda(k)+\mu(k-1)}. \tag{4.21}$$

Subtracting θ^* from both sides of (4.5) and using (4.17) and (4.18), we have

$$\tilde{\theta}_o(k) = \tilde{\theta}(k-1) + \alpha(k)P(k)w(k-1)\bar{e}'(k). \tag{4.22}$$

Now consider (3.8) with θ and $\delta(k)$ replaced by θ^* and $\delta^*(k)$, respectively. Note that θ^* is the tuned parameter matrix and $\delta^*(k)$ is the plant uncertainty matrix for θ^*. To use simple notations, we will regard θ and $\delta(k)$ as θ^* and $\delta^*(k)$. Inserting (3.8) into (4.1) gives

$$e(k) = -\tilde{\theta}'(k-1)\phi(k-1)) + \delta'(k)\phi(k-1) + d(k). \tag{4.23}$$

As (4.2), we define a normalized variable $\bar{d}(k)$ for $d(k)$ by

$$\bar{d}(k) = d(k)/n(k-1) \tag{4.24}$$

where $n(k)$ is as in (4.3). Then it follows from (4.23) and (4.24) that

$$\bar{e}(k) = -\tilde{\theta}'(k-1)w(k-1) + \eta(k) \tag{4.25}$$

where

$$\eta(k) = \delta'(k)w(k-1) + \bar{d}(k). \tag{4.26}$$

Combining (4.22) and (4.25) gives

$$\begin{aligned}\tilde{\theta}_o(k) = \{I - \alpha(k)P(k)w(k-1)w'(k-1)\}\tilde{\theta}(k-1) \\ + \alpha(k)P(k)w(k-1)\eta'(k).\end{aligned} \tag{4.27}$$

On the other hand, postmultiplying $\lambda(k)P^{-1}(k-1)$ on both sides of (4.6), and using (4.21), we have

$$\lambda(k)P(k)P^{-1}(k-1) = I - \alpha(k)P(k)w(k-1)w'(k-1). \tag{4.28}$$

Thus, we obtain from (4.27)

$$\tilde{\theta}_o(k) = \lambda(k)P(k)P^{-1}(k-1)\tilde{\theta}(k-1) + \alpha(k)P(k)w(k-1)\eta'(k) \tag{4.29}$$

which gives

$$P^{-1}(k)\tilde{\theta}_o(k) = \lambda(k)P^{-1}(k-1)\tilde{\theta}(k-1) + \alpha(k)w(k-1)\eta'(k). \tag{4.30}$$

Let

$$v_o(k) = \text{tr}\{\tilde{\theta}_o'(k)P^{-1}(k)\tilde{\theta}_o(k)\}. \tag{4.31}$$

Inserting (4.22) and (4.30) into (4.31), we get

$$\begin{aligned} v_o(k) &= \lambda(k)v(k-1) + \alpha(k)\ \text{tr}\{\tilde{\theta}'(k-1)w(k-1)\eta'(k)\} \\ &\quad + \alpha(k)\frac{\lambda(k)\ \text{tr}\{\bar{e}(k)w'(k-1)\tilde{\theta}(k-1)\} + \mu(k-1)\ \text{tr}\{\bar{e}(k)\eta'(k)\}}{\lambda(k)+\mu(k-1)} \end{aligned} \tag{4.32}$$

where (4.7) and (4.21) were used. Using (4.25), we obtain from (4.32)

$$v_o(k) - \lambda(k)v(k-1) = -\alpha(k)H(k) \tag{4.33}$$

where

$$H(k) = \frac{\lambda(k)\|\bar{e}(k)\|^2}{\lambda(k)+\mu(k-1)} - \|\eta(k)\|^2. \tag{4.34}$$

Noting that $\theta(k) = \theta_o(k)$ if $\theta_o(k) \in \Omega$, we further obtain from (4.13) and (4.33)

$$\begin{aligned} v(k) - \lambda(k)v(k-1) &\le -\alpha(k)H(k) \\ &\quad - \text{tr}\{(\theta_o(k)-\theta(k))'P^{-1}(k)(\theta_o(k)-\theta(k))\}. \end{aligned} \tag{4.35}$$

Since $\sup\{\|d(k)\|\} \leq W$ and $\sup\{\|\delta(k)\|\} \leq D$ by hypothesis and $\bar{d}(k) = d(k)/n(k-1)$ from (4.24), the equation (4.26) gives

$$\|\eta(k)\| \leq \|w(k-1)\|D + W/n(k-1).$$

Therefore,

$$H^*(k) \leq H(k) \tag{4.36}$$

where

$$H^*(k) = \frac{\lambda(k)\|\bar{e}(k)\|^2}{\lambda(k)+\mu(k-1)} - (\|w(k-1)\|D + W/n(k-1))^2. \tag{4.37}$$

On the other hand, from (4.16) and (4.37), we have

$$H^*(k) = \frac{\lambda(k)}{\lambda(k)+\mu(k-1)}(\|\bar{e}(k)\|^2 - \Delta(k)^2). \tag{4.38}$$

It then follows from the definition of $\alpha(k)$

$$0 \leq \alpha(k)H^*(k) \;\; \forall k \tag{4.39}$$

which, together with (4.36), gives

$$0 \leq \alpha(k)H^*(k) \leq \alpha(k)H(k). \tag{4.40}$$

Therefore, we have from (4.35) and (4.40)

$$\begin{aligned} v(k) - \lambda(k)v(k-1) &\leq -\alpha(k)H^*(k) - \mathrm{tr}\{(\theta_o(k) - \theta(k))'P^{-1}(k)(\theta_o(k) - \theta(k))\} \\ &\leq 0 \end{aligned} \tag{4.41}$$

which implies that $v(k)$ is monotonically decreasing. Since $v(k) \geq 0$ for any k, it immediately follows that $v(k)$ converges. ■

4.4 Forgetting Factor $\lambda(k)$

In order for the estimation algorithm to track time-varying parameters, forgetting factors are used. A simple way is to discount the old data exponentially. Suppose that a single-output plant model is given by

$$y(k) = \theta' \phi(k-1)$$

and thus the prediction error equation is written as

$$e(k) = y(k) - \theta'(k-1)\phi(k-1)$$

where $\theta(k)$ is the parameter estimate at time k. Then, this idea is formulated as a problem to minimize the following performance index:

$$J_k = (\theta - \theta(0))'\lambda^k P^{-1}(0)(\theta - \theta(0)) + \sum_{i=1}^{k} \lambda^{k-i}(y(i) - \theta'\phi(i-1))^2$$

where λ is a constant with $0 < \lambda < 1$. The solution of this problem leads to the standard least squares algorithm with a constant forgetting factor. This simple exponential discounting works well if the plant is properly excited. However, when the excitation is not sufficiently rich, this forgetting factor may cause a problem called "estimator windup" [30]. If the regression vector is small in magnitude, $P(k)$ grows exponentially. Therefore, the estimator gain becomes large which makes the system unstable. Then, the plant will be well excited, so the regression vector will become small.

To avoid estimator windup, many kinds of variable forgetting factors have been suggested. A simple example is of the form [32]

$$\lambda(k) = a\lambda(k-1) + (1-a)$$

with $0 < a < 1$. Another example is the constant trace algorithm [16]. In this algorithm, forgetting factors are selected such that the trace of the covariance matrix is always constant. Ydstie and Sargent [39] proposed a forgetting factor based on the *Constant information principle.* The constant information principle, first introduced by Fortescue [40], is an assumption that at each step the amount of forgetting corresponds to the amount of new information in the latest measurement so that the estimation is always based on the same amount of information.

We will follow Ydstie and Sargent's constant information principle, because it is based on the reasonable assumption as well as it provides some nice properties which are very useful to prove the closed-loop stability. The key property is that the infinite product of forgetting factors is uniformly bounded below by some positive constant. It follows from this property that the covariance matrix $P(k)$ and the estimates are uniformly bounded and converge [39]. We will further extend this concept such that it can cover the plant model which has plant uncertainties or external disturbances.

We now briefly review Ydstie and Sargent's work. The presentation here will differ from that of Ydstie and Sargent by considering multi-output systems with signal normalization, while their original work was for single output systems without signal normalization. Consider the following multi-output plant model with signals normalized as in (4.2):

$$\bar{y}(k) = \theta' w(k-1). \tag{4.42}$$

The prediction error equation is then given by

$$\bar{e}(k) = \bar{y}(k) - \theta'(k-1)w(k-1). \tag{4.43}$$

Define the performance index J_k as

$$J_k = \mathrm{tr}\{(\theta - \theta(0))'\sigma(0,k)P^{-1}(0)(\theta - \theta(0))\}$$

$$+ \sum_{i=1}^{k} \sigma(i,k)\|\bar{y}(i) - \theta' w(i-1)\|^2 \tag{4.44}$$

where

$$\sigma(i,k) = \lambda(i+1)\lambda(i+2)\cdots\lambda(k) \text{ for } i < k\,;\ \sigma(k,k) = 1. \tag{4.45}$$

By some manipulation of (4.44), we obtain

$$J_k = \mathrm{tr}\{(\theta - \theta(k))'P^{-1}(k)(\theta - \theta(k))\} - \mathrm{tr}\{\theta(k)'P^{-1}(k)\theta(k))\}$$

$$+ \sigma(0,k)\mathrm{tr}\{\theta'(0)P^{-1}(0)\theta(0))\} + \sum_{i=1}^{k} \sigma(i,k)\|\bar{y}(i)\|^2 \tag{4.46}$$

where

$$P^{-1}(k) = \sigma(0,k)P^{-1}(0) + \sum_{i=1}^{k} \sigma(i,k)w(i-1)w'(i-1) \tag{4.47}$$

$$\theta(k) = P(k)\{\sigma(0,k)P^{-1}(0)\theta(0) + \sum_{i=1}^{k} \sigma(i,k)w(i-1)\bar{y}'(i)\}. \tag{4.48}$$

Equation (4.47) can be written in the recursive form

$$P^{-1}(k) = \lambda(k)P^{-1}(k-1) + w(k-1)w'(k-1). \tag{4.49}$$

On the other hand, using (4.42) we have from (4.48)

$$\theta(k) = P(k)w(k-1)\bar{y}'(k) + P(k)\lambda(k)P^{-1}(k-1)\theta(k-1). \tag{4.50}$$

It then follows from (4.43) and (4.50) that

$$\theta(k) = P(k)w(k-1)(\bar{e}(k) + \theta'(k-1)w(k-1))'$$

$$+ \lambda(k)P(k)P^{-1}(k-1)\theta(k-1). \tag{4.51}$$

Postmultiplying $P(k)$ on both sides of (4.49) yields

$$I = \lambda(k)P(k)P^{-1}(k-1) + P(k)w(k-1)w'(k-1). \tag{4.52}$$

Inserting (4.52) into (4.51) gives

$$\theta(k) \;=\; \theta(k-1) + P(k)w(k-1)\bar{e}'(k). \tag{4.53}$$

Equations (4.49) and (4.53) is the standard least squares algorithm with a forgetting factor.

Now, let $N(k)$ denote the value of J_k for $\theta(k)$. Then, we have from (4.46)

$$\begin{aligned} N(k) &= \sigma(0,k)\ \mathrm{tr}\{\theta'(0)P^{-1}(0)\theta(0)\} \\ &\quad + \sum_{i=1}^{k} \sigma(i,k)\ \|\bar{y}(i)\|^2 - \mathrm{tr}\{\theta'(k)P^{-1}(k)\theta(k)\} \end{aligned} \tag{4.54}$$

which, together with (4.53), gives

$$\begin{aligned} N(k) &= \|\bar{y}(k)\|^2 - \mathrm{tr}\{\theta(k-1)'P^{-1}(k)\theta(k-1)\} \\ &\quad - 2\,w'(k-1)\theta(k-1)\bar{e}(k) - \|\bar{e}(k)\|^2 w'(k-1)P(k)w(k-1) \\ &\quad + \lambda(k)\left(\sigma(0,k-1)\mathrm{tr}\{\theta(0)'P^{-1}(0)\theta(0))\} + \sum_{i=1}^{k-1} \sigma(i,k-1)\ \|\bar{y}(i)\|^2\right). \end{aligned} \tag{4.55}$$

Using (4.43), (4.49) and (4.54), we have from (4.55)

$$N(k) = \lambda(k)N(k-1) + \|\bar{e}(k)\|^2(1 - w'(k-1)P(k)w(k-1)). \tag{4.56}$$

By using the matrix inversion lemma, (4.49) can be written as

$$P(k)w(k-1) = \frac{P(k-1)w(k-1)}{\lambda(k) + \mu(k-1)}.$$

Therefore, noting that $\mu(k) = w'(k)P(k)w(k)$, we get from (4.56) the recursive form for $N(k)$

$$N(k) = \lambda(k)N(k-1) + \frac{\lambda(k)\|\bar{e}(k)\|^2}{\lambda(k) + \mu(k-1)}. \tag{4.57}$$

We regard $N(k)$ as a measure of information. Then setting $N(k) = N(0)$ for some $N(0) > 0$ gives a forgetting factor due to the constant information principle. To make the computation of $\lambda(k)$ easier, (4.57) can be modified to

$$N(k) = \lambda(k)N(k-1) + \frac{\lambda(k)\|\bar{e}(k)\|^2}{1+\mu(k-1)} \tag{4.58}$$

In practice, the resulting algorithm will behave almost like that of (4.57). From (4.58), we see that setting $N(k) = N(0)$ gives

$$0 < \lambda(k) \leq 1 \quad \forall k \in \mathcal{Z}^+ \tag{4.59}$$

where $\mathcal{Z}^+$ is the set of nonnegative integers. Moreover, Ydstie and Sargent [39] showed that $P(k)$ and $\theta(k)$ are uniformly bounded and

$$\lim_{k\to\infty} \lambda(k) = 1$$

when the algorithm (4.49) and (4.53) with a forgetting factor formula (4.57) or (4.58) is applied to the linear time-invariant plant without plant uncertainties or external disturbances.

However, when applied to the plant model with unmodelled plant uncertainties or external disturbances, this forgetting factor does not guarantee the boundedness of $P(k)$ or $\theta(k)$. Thus, we need to modify the measure of information $N(k)$.

Roughly speaking, if new information described by the second term on the right side of (4.58) is presumably bad, we neglect it. The decision is made by the dead zone, i.e., if $\|\bar{e}(k)\|$ is less that the size of dead zone $\Delta(k)$ at time k, the new information is neglected.

Based on this idea, we modify (4.58) as

$$N(k) = \lambda(k)N(k-1) + S(H_1^*(k))\,;\; N(0) > 0 \tag{4.60}$$

where the function $S(\cdot)$ is defined by

$$S(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.61}$$

and

$$H_1^*(k) = M_1(k)\lambda(k) - M_2(k) \tag{4.62}$$

with

$$M_1(k) = \frac{\|\bar{e}(k)\|^2}{1+\mu(k-1)}$$

$$M_2(k) = (\|w(k-1)\|D + W/n(k-1))^2$$

Now we choose $\lambda(k)$ such that

$$N(k) = N(0) \quad \forall k \in \mathcal{Z}^+ \tag{4.63}$$

where $N(0)$ is a design parameter. Then (4.60) gives

$$N(0) = \lambda(k)N(0) + S(H_1^*(k)). \tag{4.64}$$

It then follows from (4.62) and (4.64) that

$$\lambda(k) = \begin{cases} (N(0)+M_2(k))/(N(0)+M_1(k)) & \text{if } M_2(k)/M_1(k) < 1 \\ 1 & \text{otherwise} \end{cases} \tag{4.65}$$

which always satisfies $0 < \lambda(k) \leq 1$.

4.5 Properties of the Proposed Estimation Algorithm

In this section, some important properties of the proposed algorithm will be summarized in the following two lemmas.

Lemma 4.2. If $\lambda(k)$ is chosen such that it satisfies (4.64), then

i) there exists $\sigma^* > 0$ such that

$$\sigma^* \leq \sigma(0,k) \ \ \forall k \geq 0 \tag{4.66}$$

where

$$\sigma(i,k) = \lambda(i+1)\lambda(i+2)\cdots\lambda(k) \text{ for } i < k\,; \sigma(k,k) = 1. \tag{4.67}$$

ii) $$\lambda(k) \to 1, \text{ as } k \to \infty. \tag{4.68}$$

Proof. Solving (4.60) recursively, we obtain

$$N(k) = \sigma(0,k)N(0) + \sum_{i=1}^{k} \sigma(i,k)S(H_1^*(k)). \tag{4.69}$$

Now, from the definition of $\alpha(k)$ and $H^*(k)$ defined in (4.38), we have

$$\alpha(k)H^*(k) = S(H^*(k)). \tag{4.70}$$

This, together with (4.41) gives

$$\begin{aligned} v(k) \ &\leq \ \sigma(0,k)v(0) - \sum_{i=1}^{k} \sigma(i,k)S(H^*(i)) \\ &- \sum_{i=1}^{k} \sigma(i,k)\mathrm{tr}\{(\theta_o(i) - \theta(i))' P^{-1}(i)(\theta_o(i) - \theta(i))\} \end{aligned} \tag{4.71}$$

Comparing (4.62) with (4.37), we obtain

$$H_1^*(k) \leq H^*(k). \tag{4.72}$$

Since (4.72) implies that $S(H_1^*(k)) \leq S(H^*(k))$, it follows from (4.71) that

$$v(k) \leq \sigma(0,k)v(0) - \sum_{i=1}^{k} \sigma(i,k)S(H_1^*(k)). \tag{4.73}$$

From (4.69) and (4.73), noting that $N(k) = N(0)$, we have

$$\sigma(0,k) \geq N(0)/(N(0) + v(0)) \tag{4.74}$$

which implies Property i). ii) follows immediately from i). ∎

Lemma 4.3. For the parameter estimation algorithm (4.4) - (4.7) applied to the plant (3.8) with the function $\alpha(k)$ given in (4.15) where $\sup\{\|d(k)\|\} \leq W$ and $\sup\{\|\delta(k)\|\} \leq D$ and with the forgetting factor given in (4.65), the following properties hold:

i) $P(k)$ is uniformly bounded and converges.

ii) $$\limsup_{k} \{\|\bar{e}(k)\| - K(\|w(k-1)\|D + W/n(k-1))\} \leq 0 \tag{4.75}$$

where $K \geq (1 + \|P(\infty)\|)^{1/2}$ and $\lim_{k\to\infty} P(k) = P(\infty)$.

iii) $\theta(k)$ is uniformly bounded. Furthermore,

$$\|\theta(k) - \theta(k-1)\| \to 0, \quad \text{as} \quad k \to \infty \tag{4.76}$$

iv) The size of dead zone $\Delta(k)$ given by (4.16) is bounded.

Proof. By the matrix inversion lemma, (4.6) can be written as

$$P^{-1}(k) = \lambda(k)P^{-1}(k-1) + \alpha(k)w(k-1)w'(k-1) \tag{4.77}$$

which gives

$$P^{-1}(k) = \sigma(0,k)M^{-1}(k) \tag{4.78}$$

where

$$M^{-1}(k) = P^{-1}(0) + \sum_{i=1}^{k} \frac{\alpha(i)}{\sigma(0,i)} w(i-1)w'(i-1). \tag{4.79}$$

From (4.78), we have

$$\lambda_{min}(P^{-1}(k)) \geq \sigma(0,k)\lambda_{min}(P^{-1}(0)) \tag{4.80}$$

where $\lambda_{min}(A)$ denotes the minimum eigenvalue of a matrix A. It then follows from (4.74) and (4.80) that

$$\|P(k)\| \leq (1 + v(0)/N(0))\|P(0)\| \tag{4.81}$$

which implies that $P(k)$ is uniformly bounded. On the other hand, it is already known that $M(k)$ converges (see, e.g., Samson [8]) and it follows from Lemma 4.2 i) that a sequence $\{\sigma(0,k)\}$ converges and the limit $\sigma(0,\infty)$ satisfies $\sigma(0,\infty) \geq \sigma^*$ for some $\sigma^* > 0$. Thus (4.78) implies that $P(k)$ converges.

To show ii), we have from (4.71)

$$\begin{aligned} v(k) \leq \sigma(0,k)v(0) - \sigma^* \sum_{i=1}^{k} S(H^*(i)) \\ - \sigma^* \sum_{i=1}^{k} \mathrm{tr}\{(\theta_o(i) - \theta(i))' P^{-1}(i)(\theta_o(i) - \theta(i))\} \end{aligned} \tag{4.82}$$

where σ^* is as in (4.66). Since $v(k)$ converges due to Lemma 4.1 and for any $i \in \mathcal{Z}^+$ $S(H^*(i)) \geq 0$, $\mathrm{tr}\{(\theta_o(i) - \theta(i))' P^{-1}(i)(\theta_o(i) - \theta(i))\} \geq 0$, it follows from (4.82) that

$$S(H^*(i)) \to 0, \quad \text{as} \quad i \to \infty \tag{4.83}$$

$$\mathrm{tr}\{(\theta_o(i) - \theta(i))' P^{-1}(i)(\theta_o(i) - \theta(i))\} \to 0, \quad \text{as} \quad i \to \infty. \tag{4.84}$$

Combining (4.83) and (4.37) gives

$$\limsup_{k} \{\frac{\lambda(k)\|\bar{e}(k)\|^2}{\lambda(k) + \mu(k-1)} - (\|w(k-1)\|D + W/n(k-1))^2\} \leq 0. \tag{4.85}$$

Since $\|w(k)\| \leq 1$ and $n(k) \geq 1$ for any k, $(\|w(k-1)\|D + W/n(k-1))$ is bounded. Thus, using the facts that $\mu(k) = w'(k)P(k)w(k) \leq \|P(k)\|$ and $\lambda(k) \to 1$ as

$k \to \infty$, we obtain from (4.85)

$$\limsup_k \{\|\bar{e}(k)\|^2 - (1 + \|P(\infty)\|)(\|w(k-1)\|D + W/n(k-1))^2\} \leq 0. \tag{4.86}$$

Now, ii) readily follows (4.86).

Since $\theta(k)$ is always in Ω and Ω is bounded, $\theta(k)$ is uniformly bounded. To prove that $\|\theta(k) - \theta(k-1)\|$ converges to zero, we first show that

$$\alpha(k)P(k)w(k-1) \to 0, \text{ as } k \to \infty. \tag{4.87}$$

From (4.6), we have

$$P(k) = \frac{1}{\sigma(0,k)}\{P(0) - \sum_{i=1}^{k}\sigma(0,i)\frac{\alpha(i)P(i-1)w(i-1)w'(i-1)P(i-1)}{\lambda(i) + \mu(i-1)}\}. \tag{4.88}$$

Noting that $P(k)$ and $\sigma(0,k)$ converge and that $\sigma(0,i) \geq \sigma^*$ for some $\sigma^* > 0$, we have from (4.88)

$$\alpha(k)\frac{P(k-1)w(k-1)w'(k-1)P(k-1)}{\lambda(k) + \mu(k-1)} \to 0, \text{ as } k \to \infty. \tag{4.89}$$

Since $\lambda(k) \to 1$ as $k \to \infty$, for given ε with $0 \leq \varepsilon < 1$ there exists a positive integer T such that

$$\lambda(k) \geq 1 - \varepsilon \quad \text{for } k \geq T.$$

Thus, using (4.20) and (4.21) we have

$$\begin{aligned}
&\alpha^2(k)P(k)w(k-1)w'(k-1)P(k) \\
&\quad = \alpha(k)\frac{P(k-1)w(k-1)w'(k-1)P(k-1)}{\lambda(k) + \mu(k-1)}\frac{1}{\lambda(k) + \mu(k-1)} \\
&\quad \leq \alpha(k)\frac{P(k-1)w(k-1)w'(k-1)P(k-1)}{\lambda(k) + \mu(k-1)}\frac{1}{1-\varepsilon} \quad \forall k \geq T.
\end{aligned} \tag{4.90}$$

This, together with (4.89), yields (4.87).

Since $\theta(k)$ is bounded, $\|w(k)\| \leq 1$ and $\delta(k)$, $\bar{d}(k)$ are bounded, it follows from (4.25) that $\bar{e}(k)$ is also bounded. Therefore, we have from (4.5) and (4.87)

$$\theta_o(k) - \theta(k-1) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \tag{4.91}$$

On the other hand, noting that $P(k)$ is uniformly bounded and

$$\mathrm{tr}\{(\theta_o(k) - \theta(k))' P^{-1}(k)(\theta_o(k) - \theta(k))\} \geq$$

$$\mathrm{tr}\{(\theta_o(k) - \theta(k))'(\theta_o(k) - \theta(k))\} / \lambda_{max}(P(k))$$

where $\lambda_{max}(A)$ is the maximum eigenvalue of a matrix A, we have from (4.84)

$$\theta(k) - \theta_o(k) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \tag{4.92}$$

Combining (4.91) and (4.92) gives (4.76).

Now, we show iv). Since (4.2) implies that $\|w(k)\| \leq 1$ and $P(k)$ is uniformly bounded, $\mu(k)$ given by

$$\mu(k) = w'(k)P(k)w(k)$$

is uniformly bounded. On the other hand, Lemma 4.2 i) implies that $\lambda(k)$ is bounded below by σ^* and (4.3) gives $n(k-1) \geq 1$. Therefore, it follows from (4.16) that $\Delta(k)$ is bounded. ■

Note that the boundedness of $\Delta(k)$ is due to the signal normalization which makes $\mu(k)$ bounded. Lemma 4.3 also holds for the estimation algorithm without projection. The proof is almost same as Lemma 4.3 except for the uniform boundedness of $\theta(k)$. The uniform boundedness of $\theta(k)$ for the algorithm without projection comes from the fact that $v(k)$ as defined in (4.19) is monotonically decreasing. Since

$$v(k) \leq v(0)$$

the definition of $v(k)$ gives

$$\|\tilde{\theta}'(k)\tilde{\theta}(k)\|\lambda_{min}(P^{-1}(k)) \leq v(0). \tag{4.93}$$

Using (4.81), we obtain from (4.93)

$$\|\tilde{\theta}'(k)\tilde{\theta}(k)\| \leq v(0)\|P(0)\|(1 + v(0)/N(0)) \tag{4.94}$$

which implies that $\theta(k)$ is uniformly bounded.

Remark 4.1. The equation (4.76) in Lemma 4.3 does not imply that $\theta(k)$ is a convergent sequence. For example, consider a sequence of real numbers

$$x(k+1) = x(k) + v(k)$$

where

$$\{v(k)\} = \{\ 1 \ \underbrace{-\frac{1}{2}\cdots-\frac{1}{2}}_{4}\ \underbrace{\frac{1}{3}\cdots\frac{1}{3}}_{6}\ \underbrace{-\frac{1}{4}\cdots-\frac{1}{4}}_{8}\cdots\ \}.$$

Even though $v(k)$ converges to zero, $x(k)$ does not converge.

CHAPTER FIVE
ADAPTIVE CONTROLLER

In this chapter, we construct a controller which makes the estimated model exponentially stable. Basically, we follow Samson and Fuchs' idea [7] to construct the controller. The controller, suggested by Samson and Fuchs, consists of a state observer for the plant and a state feedback control law for the estimated model. The observer gain was selected such that all observer poles were located in the origin and the control law was designed such that it made the estimated model exponentially stable. With this controller structure, they showed that the closed-loop system for the linear time-invariant plant can be made exponentially stable, in the absence of the plant uncertainties or the external disturbances.

We will construct a controller with the same philosophy as Samson and Fuchs [7], but in the different setting, i.e., for linear time-varying plants which may be fast time-varying with plant uncertainties and external disturbances. Moreover, we will use a different state feedback control law based on Kleinman's method (see Kamen and Khargonekar [41] for the linear time-invariant system, and Moore and Anderson [20] or Cheng [42] for the linear time-varying system) to make the estimated model exponentially stable. Its advantage over other methods such as the LQ (Linear Quadratic) control law [43], the pole placement and the transfer function approach [36, 45], is ease of implementation, especially when the reachability index N is greater than the system order.

We assume that $A_p(k)$ is invertible and therefore the plant model with $p < s$ has a realization with the same order as the plant model with $p \geq s$ (see Section 3.1). Furthermore, we assume that $p \geq s$ and the plant model of the form (3.11) and (3.12). However, the argument in this chapter can be also applied to the plant model with $p < s$ *mutatis mutandis*.

Consider the regression model (3.28). Since the model depends on θ in Ω, its observable realization can be viewed as a function of θ. Thus, to indicate the dependency of θ, we will denote a system matrix $F(k)$ and an input matrix $G(k)$ of (3.28) by $F(\theta, k)$ and $G(\theta, k)$, respectively. Further, substituting $\theta(k)$ for θ will give the observable realization $(F(\theta, k), G(\theta, k))$ of the estimated model.

5.1 Adaptive State Observer

Let an $n \times 1$ vector $\hat{x}(k)$ denote the estimated state vector of the plant. Then the adaptive state observer is given by

$$\hat{x}(k+1) = F(\theta(k), k)\hat{x}(k) + G(\theta(k), k)u(k) + M(k)v(k) \tag{5.1}$$

where

$$v(k) = y(k) - H\hat{x}(k) \tag{5.2}$$

$$F(\theta(k), k) = \begin{bmatrix} A_1(\theta(k), k+1) & I & 0 \cdots 0 \\ A_2(\theta(k), k+2) & 0 & I \cdots 0 \\ \vdots & & \\ A_{p-1}(\theta(k), k+p-1) & 0 & 0 \cdots I \\ A_p(\theta(k), k+p) & 0 & 0 \cdots 0 \end{bmatrix} \tag{5.3}$$

$$G(\theta(k), k) = \begin{bmatrix} B_1(\theta(k), k+1) \\ B_2(\theta(k), k+2) \\ \vdots \\ B_{p-1}(\theta(k), k+p-1) \\ B_p(\theta(k), k+p) \end{bmatrix} \tag{5.4}$$

and

$$H = [\, I \; 0 \cdots 0 \,]. \tag{5.5}$$

Note that $B_i(\theta(k), k+i) = 0$ if $i > s$. Associated with (3.3)-(3.4), $A_i(\theta(k), k+i)$, $i = 1, ..., p$ in (5.3) and $B_i(\theta(k), k+i)$, $i = 1, ..., s$ in (5.4) can be written as

$$A_i(\theta(k), k+i) = \sum_{j=1}^{r} A_{ij}(k) f_j(k+i), \; i = 1, ..., p \tag{5.6}$$

$$B_i(\theta(k), k+i) = \sum_{j=1}^{r} B_{ij}(k) f_j(k+i), \; i = 1, ..., s \tag{5.7}$$

where $A_{ij}(k)$ and $B_{ij}(k)$ are submatrices of $\theta'(k)$ satisfying

$$\theta'(k) \;\; = \;\; [\, A_{11}(k) \cdots A_{1r}(k) \cdots A_{p1}(k) \cdots A_{pr}(k) \; B_{11}(k) \cdots B_{sr}(k) \,]. \tag{5.8}$$

We select $M(k)$ such that all of the observer poles are located in the origin, i.e.,

$$M'(k) = [\, A_1'(\theta(k), k+1) \; A_2'(\theta(k), k+2) \; \cdots \; A_p'(\theta(k), k+p) \,]. \tag{5.9}$$

This type of observer has been frequently used, see for example, Samson [8], Ossman and Kamen [13], and Kreisselmeir [29] in the more general form.

5.2 Kleinman's Method

When a pair $(F(k), G(k))$ is uniformly reachable in N steps, the feedback control law according to Kleinman's method [20] is given by

$$u(k) = -L(k)x(k) \tag{5.10}$$

where

$$L(k) = G'(k)\Phi'(k+N+1, k+1)Y_{N+1}^{-1}(k)\Phi(k+N+1, k+1)F(k) \qquad (5.11)$$

and $Y_N(k)$ is the N-step reachability grammian defined in (3.25). Note that $Y_{N+1}(k)$ was used instead of $Y_N(k)$. The validity of Kleinman's method is based on the following theorem [20].

Theorem 5.1. With $(F(k), G(k))$ uniformly reachable in N steps, $(F(k), G(k))$ bounded above, and the state feedback gain $L(k)$ defined in (5.11), the closed loop system

$$x(k+1) = (F(k) - G(k)L(k))x(k)$$

is exponentially stable.

5.3 Properties of the Estimated Model as a Sequence

Corresponding to the notation introduced for $F(\theta, k)$ and $G(\theta, k)$, we will use $L(\theta, k)$ and $Y_{N+1}(\theta, k)$ instead of $L(k)$ and $Y_{N+1}(k)$ to indicate the dependency on θ. It is useful to consider two families $\{F(\theta, k)\}$ and $\{G(\theta, k)\}$ of continuous functions on Ω where each of $F(\theta, k)$ and $G(\theta, k)$ with a fixed k is a member. Then we have the following lemma.

Lemma 5.1. $\{F(\theta, k)\}$ and $\{G(\theta, k)\}$ are equicontinuous and uniformly bounded on Ω.

Remark 5.1. A family $\mathcal{F}$ whose member is a continuous function defined on Ω is said to be equicontinuous on Ω if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $f \in \mathcal{F}$ $\|f(x) - f(y)\| < \varepsilon$ whenever $\|x - y\| < \delta$, $x \in \Omega$ and $y \in \Omega$ [46].

<u>Proof of Lemma 5.1</u>. It suffices to prove only for $\{G(\theta,k)\}$, since the same argument can be used for $\{F(\theta,k)\}$.

Replacing $\theta(k)$ in (5.7) by θ defined as (3.5), we have

$$B_i(\theta,k+i)=\sum_{j=1}^{r} B_{ij}f_j(k+i),\ i=1,...,s \tag{5.12}$$

Then (5.4) gives

$$\begin{aligned}\|G(\theta,k)\|^2 &= \sum_{i=1}^{s}\|\sum_{j=1}^{r}B_{ij}f_j(k+i)\|^2\\ &\leq \sum_{i=1}^{s}\sum_{j=1}^{r}|B_{ij}|^2\,|f_j(k+i)|^2\end{aligned} \tag{5.13}$$

Since $f_j(k),\ j=1,...,r$ are bounded functions, there is $M>0$ such that

$$|f_j(k)|<M\ \ \forall k\in\mathcal{Z}^+,\ j=1,...,r. \tag{5.14}$$

Let A be an $n_1\times n_2$ matrix. Then the Frobenius norm [47] $\|A\|_F$ of A defined as

$$\|A\|_F=\left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}|a_{ij}|^2\right)^{1/2}$$

satisfies

$$\|A\|\leq\|A\|_F\leq\sqrt{n_1}\,\|A\| \tag{5.15}$$

where a_{ij} is the ij-th element of A and $\|A\|$ is the Euclidean norm of A.

Further, we define a $m\times sr$ submatrix B of θ' by

$$B=[\,B_{11}\cdots B_{1r}\cdots B_{p1}\cdots B_{sr}\,] \tag{5.16}$$

Using (5.14), (5.15) and (5.16), we obtain from (5.13)

$$\begin{aligned}\|G(\theta,k)\|^2 &\leq M^2m\|B\|^2\\ &\leq M^2m\|\theta\|^2.\end{aligned} \tag{5.17}$$

Since Ω is compact (thus bounded), uniform boundedness of $\{G(\theta, k)\}$ immediately follows from (5.17).

For given $\varepsilon > 0$, let $\delta = \varepsilon/(M\sqrt{m})$. Then, from the definition of $G(\theta, k)$ and (5.17), we get

$$\|G(x,k) - G(y,k)\| = \|G(x-y,k)\| < \varepsilon$$

for all x and y in Ω for which $\|x - y\| < \delta$, which implies that $\{G(\theta, k)\}$ is equicontinuous on Ω. ■

Lemma 5.2. Suppose that $\{f_n(x) : \Omega \to \Pi,\ n \in \mathcal{Z}^+\}$ is equicontinuous on Ω, φ is uniformly continuous on Π, and $h_n(x) = \varphi(f_n(x))$. Then $\{h_n(x)\}$ is equicontinuous on Ω. Moreover, if $\{f_n(x)\}$ is uniformly bounded on Ω, then $\{h_n(x)\}$ is also uniformly bounded on Ω.

The froof of Lemma 5.2 follows directly from the definitions.

Lemma 5.3. $\{Y_{N+1}(\theta, k)\}$ is equicontinuous and uniformly bounded on Ω.

Proof. Define $g_k(\theta)$ on Ω by

$$g_k(\theta) = [\,F(\theta,k)\cdots F(\theta,k+N-1)\ G(\theta,k)\cdots G(\theta,k+N-1)\,]$$

and choose a compact set Π such that

$$g_k(\theta) \in \Pi \ \text{ for } \theta \in \Omega,\ k \in \mathcal{Z}^+.$$

Such a compact set always exists because $\{g_k(\theta)\}$ is uniformly bounded on Ω by Lemma 5.1. From the definition of $Y_{N+1}(\theta, k)$ as in (3.25), we see that any element of $Y_{N+1}(\theta, k)$ is a uniformly continuous function on Π. Then the result immediately follows from Lemma 5.2. ■

Corollary. $\{\det(Y_{N+1}(\theta,k))\}$ and $\{\mathrm{adj}(Y_{N+1}(\theta,k))\}$ are equicontinuous and uniformly bounded on Ω, where $\mathrm{adj}(Y_{N+1}(\theta,k))$ is the adjoint matrix of $Y_{N+1}(\theta,k)$.

Lemma 5.4. $\{Y_{N+1}^{-1}(\theta,k)\}$ is equicontinuous and uniformly bounded on Ω.

Proof. It follows from Assumption 6 in Section 3.1 that $Y_{N+1}^{-1}(\theta,k)$ always exists for any $k \in \mathcal{Z}^+$, any $\theta \in \Omega$. The remainder of the proof follows from Lemma 5.2 and Corollary of the Lemma 5.3. ■

Corollary. $\{L(\theta,k)\}$ and $\{F(\theta,k) - G(\theta,k)L(\theta,k)\}$ are equicontinuous and uniformly bounded on Ω.

5.4 Closed-Loop Stability of the Estimated Model

In this section, it will be shown that we can construct a feedback control law based on Kleinman's method which makes the estimated model exponentially stable. Let $\hat{Y}_{N+1}(\theta(k),\cdots,\theta(k+N),k)$ and $\hat{L}(\theta(k),\cdots,\theta(k+N),k)$ be the $(N+1)$-step reachability grammian and Kleinman's feedback gain, respectively, at time k for the system $(F(\theta(k),k),G(\theta(k),k))$. Note that the $(N+1)$-step reachability grammian and Kleinman's feedback gain at time k are functions of $\theta(k+1),\cdots,\theta(k+N)$ as well as of $\theta(k)$.

Since we do not know the future parameter estimates, we will use the current estimate $\theta(k)$ instead of $\theta(k+1),\cdots,\theta(k+N)$ to calculate the $(N+1)$-step reachability grammian and Kleinman's feedback gain of the estimated model at time k. The resulting reachability grammian will be $\hat{Y}_{N+1}(\theta(k),\cdots,\theta(k),k)$ or simply $Y_{N+1}(\theta(k),k)$ which comes from the notation $Y_{N+1}(\theta,k)$ already introduced, and Kleinman's feedback gain will be $\hat{L}(\theta(k),\cdots,\theta(k),k)$ or simply $L(\theta(k),k)$.

Now, consider $\hat{Y}_{N+1}(\theta_1,\cdots,\theta_{N+1},k)$ and $\hat{L}(\theta_1,\cdots,\theta_{N+1},k)$ as functions defined on Ω^{N+1} $\forall\theta_1,\cdots,\theta_{N+1} \in \Omega$. Then, these functions are also equicontinuous and uniformly bounded on Ω (which can be proved in a similar way to Lemma 5.3). Using these notations and facts, we have the following theorem.

Theorem 5.2. With the parameter estimation algorithm introduced in Chapter Four, let $\theta(k)$ be the estimated parameter matrix at k-th instant. Then there exists a bounded sequence of feedback gains $\{L(\theta(k), k))\}$ such that

$$(F(\theta(k), k) - G(\theta(k), k)L(\theta(k), k))$$

is exponentially stable.

Proof. By Corollary of Lemma 5.4, $\{L(\theta, k)\}$ obtained from Kleinman's method is equicontinuous and uniformly bounded on Ω. Since $\theta(k) \in \Omega$ for any $k \in \mathcal{Z}^+$, it follows that $\{L(\theta(k), k)\}$ is bounded.

Now, by Assumption 6 in Section 3.1, there is $\varepsilon > 0$ such that

$$|\det(Y_N(\theta, k))| \geq \varepsilon \ \ \forall k \in \mathcal{Z}^+, \ \ \forall \theta \in \Omega. \tag{5.18}$$

Moreover, since $\det(\hat{Y}_{N+1}(\theta_1, \cdots, \theta_{N+1}, k))$ is equicontinuous on Ω (which can be proved in a similar way to Lemma 5.3), for given $\varepsilon_2 > 0$ with $\varepsilon_2 < \varepsilon$ there exists $T \in \mathcal{Z}^+$ such that for any $k \geq T$, for any $\theta \in \Omega$

$$|\det(Y_N(\theta, k))| - |\det(\hat{Y}_{N+1}(\theta(k), \cdots, \theta(k+N), k))| \leq \varepsilon_2. \tag{5.19}$$

Combining (5.18) and (5.19) gives for any $k \geq T$

$$0 < \varepsilon - \varepsilon_2 \leq |\det(\hat{Y}_{N+1}(\theta(k), \cdots, \theta(k+N), k))|$$

which implies that the estimated model is uniformly reachable in N steps for $k \geq T$.

Therefore, $(F(\theta(k), k) - G(\theta(k), k))\hat{L}(\theta(k), \cdots, \theta(k+N), k))$ is exponentially stable and thus for given δ with $0 < \delta < 1$, there exists $t \in \mathcal{Z}^+$ such that

$$\|J_t(k)\| < \delta/2 \ \ \forall k \geq T \tag{5.20}$$

where

$$J_t(k) = \prod_{i=1}^{t}(F(\theta(k+i), k+i) -$$

$$G(\theta(k+i), k+i)\hat{L}(\theta(k+i), \cdots, \theta(k+N+i), k+i)). \qquad (5.21)$$

Since $F(\theta, k)$ and $G(\theta, k)$ are uniformly bounded on Ω, $\hat{L}(\theta(k), \cdots, \theta(k+N), k)$ is equicontinuous on Ω, and $\|\theta(k) - \theta(k-1)\|$ converges to zero, there is $T^* \in \mathcal{Z}^+$ with $T^* \geq T$ such that for any $k \geq T^*$

$$\|J_t(k) - \prod_{i=1}^{t}(F(\theta(k+i), k+i)$$

$$- G(\theta(k+i), k+i)L(\theta(k+i), k+i))\| < \delta/2. \qquad (5.22)$$

It then follows from (5.20) and (5.22) that for any $k \geq T^*$

$$\|\prod_{i=1}^{t}(F(\theta(k+i), k+i) - G(\theta(k+i), k+i)L(\theta(k+i), k+i)\| < \delta.$$

which completes the proof. ■

5.5 Adaptive Control Law

We now define an adaptive control law by

$$u(k) = -L(\theta(k), k)\hat{x}(k) + r(k) \qquad (5.23)$$

where $\hat{x}(k)$ is the estimated state vector given in (5.1), $L(\theta(k), k)$ is the feedback gain obtained from Kleinman's method and $r(k)$ is the reference signal. Besides the control objective mentioned in Section 3.2, some other specifications can be given. One of the important specifications is to require an output, say $y_1(k)$, of the plant to track the reference signal $r(k)$ so that the track error

$$|y_1(k) - r(k)|$$

is suitably small. To meet this specification, we can use the internal model principle. The design procedure is as follows. First, we realize the internal model from the dynamics of the reference signal and external disturbances. Next, we augment the plant model with the internal model. Finally, we design a control law such that the augmented model is asymptotically stable.

Suppose that $r(k)$ is a step command and

$$y_1(k) = h\ y(k)$$

where an $1 \times m$ vector h is given by

$$h = [\,1\ 0\ \cdots\ 0\,].$$

The internal model is then realized as

$$z(k+1) = z(k) + (h\ y(k) - r(k)). \tag{5.24}$$

Combining the state observer (5.1) - (5.5) and the internal model (5.24) gives

$$\begin{aligned} \begin{bmatrix} \hat{x}(k+1) \\ z(k+1) \end{bmatrix} &= F_a(\theta(k), k) \begin{bmatrix} \hat{x}(k) \\ z(k) \end{bmatrix} + G_a(\theta(k), k)u(k) \\ &\quad + E_1 r(k) + E_2(k)v(k) \end{aligned} \tag{5.25}$$

where

$$F_a(\theta(k), k) = \begin{bmatrix} F(\theta(k), k) & O_{n\times 1} \\ h\,H & 1 \end{bmatrix} \;;\; E_1 = \begin{bmatrix} O_{n\times 1} \\ -1 \end{bmatrix}$$

$$G_a(\theta(k), k) = \begin{bmatrix} G(\theta(k), k) \\ 0 \end{bmatrix} \;;\; E_2(k) = \begin{bmatrix} M(k) \\ h \end{bmatrix}$$

and $O_{r\times s}$ denotes an $r \times s$ zero matrix. The augmented estimated model is then denoted by a pair $(F_a(\theta(k), k), G_a(\theta(k), k))$. For this system, Kleinman's method

gives the feedback gain $L_a(\theta(k),k)$ and with this feedback gain we construct the following control law:

$$u(k) = -L_a(\theta(k),k)\begin{bmatrix} \hat{x}(k) \\ z(k) \end{bmatrix}. \tag{5.26}$$

Suppose that the augmented estimated model given in (5.25) satisfies the following assumption.

Assumption 6.a: There is $\varepsilon > 0$ such that for any $\theta \in \Omega$ where Ω is as defined in (3.27), the augmented model $(F_a(\theta,k), G_a(\theta,k))$ satisfies

$$|\det(Y_N(\theta,k))| \geq \varepsilon \;\; \forall k \in \mathcal{Z}^+$$

where $Y_N(\theta,k)$ is its reachability grammian, for some $N \in \mathcal{Z}^+$.
Then, Theorem 5.2 can be applied to this augmented estimated model. However, Assumption 6 does not imply Assumption 6.a, so the control law (5.26) may result in the instability of the closed-loop system $(F_a(\theta(k),k) - G_a(\theta(k),k)L_a(\theta(k),k))$. To avoid this, we suggest the following two approaches. The first approach is to take a set Ω such that it satisfies Assumption 6.a in this section instead of Assumption 6 in Section 3.1. The second approach is to switch the control law (5.26) to that of the estimated model without augmentation when the augmented model approaches to a system which is not reachable. As for the augmented model with the internal model (5.24), this can occur when a transmission zero of the estimated model approaches 1.

Remark 5.2. The transfer function theories for linear time-varying systems are based on a non-commutative ring structure called the skew ring of polynomials. However, poles and zeros can be defined in a very similar manner to the linear time-invariant systems. See [36, 45] for the transfer function approach to linear

time-varying systems, and [48] for the definition of poles and zeros of linear time-varying systems.

If control laws are switched, the output may not track the reference signal very well. However, since the estimated model satisfies Assumption 6 in Section 3.1 stability of the closed-loop system $(F(\theta(k), k) - G(\theta(k), k)L(\theta(k), k))$ is guaranteed. Although the internal model presented in this example is a simple one, the same principle applies in more general cases.

CHAPTER SIX
ROBUST STABILITY

In this chapter, we show that the control law (5.23) and an adaptive state observer (5.1), all based on the latest estimates $\theta(k)$, make the closed-loop system robustly globally stable. To prove this, we need the following:

- A bounded sequence of feedback gains which makes the estimated model exponentially stable.
- An estimation algorithm in which $\|\theta(k) - \theta(k-1)\|$ converges to zero and Lemma 4.3 ii) is satisfied.

In fact, for any adaptive controller we can use the arguments of this chapter to prove the robust stability of the closed-loop system if the above two features hold. Therefore, the arguments in this chapter are also valid for the control law based on the internal model principle, e.g., (5.26) *mutatis mutandis.*

As defined in Section 3.2, robust stability means that there exists $D^* > 0$ such that for any bounded external disturbances $d(k)$ and for all possible plant uncertainties $\delta^*(k)$ which satisfy

$$\sup\{\|\delta^*(k)\|\} \leq D^*$$

the closed-loop system is stable.

<u>Theorem 6.1</u>. There exists $D^* > 0$ such that for any D and W which are design parameters used in calculating the size of dead zone with $0 \leq D < D^*$, and for any

bounded external disturbances $d(k)$ and plant uncertainties $\delta^*(k)$ satisfying

$$\sup\{\|\delta^*(k)\|\} \leq D$$

$$\sup\{\|d(k)\|\} \leq W$$

the closed-loop system described by the plant (3.1), the parameter estimation algorithm (4.4)-(4.6), the state observer (5.1) and the control law (5.23) is globally stable.

Proof. From (5.1)-(5.5), we have

$$v(k) = y(k) - \sum_{i=1}^{p} A_i(\theta(k-i), k)\, y(k-i) - \sum_{i=1}^{s} B_i(\theta(k-i), k)\, u(k-i). \tag{6.1}$$

On the other hand, it follows from (4.1) together with (3.6) and (5.6)-(5.8) that

$$\begin{aligned} e(k) &= y(k) - \theta'(k-1)\phi(k-1) \\ &= y(k) - \sum_{i=1}^{p} A_i(\theta(k-1), k) y(k-i) - \sum_{i=1}^{s} B_i(\theta(k-1), k) u(k-i). \end{aligned} \tag{6.2}$$

Then (6.1) and (6.2) gives

$$v(k) = e(k) + R(k-1)\phi(k-1) \tag{6.3}$$

where

$$\begin{aligned} R(k-1) = [\, & A_{11}(k-1) - A_{11}(k-1) \cdots A_{1r}(k-1) - A_{1r}(k-1) \cdots \\ & A_{p1}(k-1) - A_{p1}(k-p) \cdots A_{pr}(k-1) - A_{pr}(k-p) \\ & B_{11}(k-1) - B_{11}(k-1) \cdots B_{sr}(k-1) - B_{sr}(k-s)\,]. \end{aligned} \tag{6.4}$$

Note that $A_{ij}(k)$ or $B_{ij}(k)$ in (6.4) is a submatrix of $\theta'(k)$. Define $\psi(k)$ as

$$\psi'(k) = [\, y'(k)\, y'(k-1) \cdots y'(k-p+1)\, u(k) \cdots u(k-s+1)\,]. \tag{6.5}$$

Then (3.6) gives

$$\phi(k) = N(k)\psi(k) \tag{6.6}$$

where the elements of $N(k)$ are either zero or one of the functions $f_1(k+1), ..., f_r(k+1)$. Since $f_j(k), j = 1, ..., r$ are bounded, for some $N^* > 0$

$$\|N(k)\| \le N^* \quad \forall k \in \mathcal{Z}^+.$$

Since $R(k) \to 0$ as $k \to \infty$ by Lemma 4.3 iii) and $n(k) \le 1 + \|\phi(k)\|$ from (4.3), Lemma 4.3 ii), together with (6.3) and (6.6), implies that for given $\varepsilon > 0$ there is $T \in \mathcal{Z}^+$ such that for any $k \ge T$

$$\|v(k)\| - (KD + \varepsilon)N^*\|\psi(k-1)\| - KW \le \varepsilon(1 + N^*\|\psi(k-1)\|). \tag{6.7}$$

Now using the state observer (5.2) and the control law (5.23), we can write (6.5) as

$$\psi(k) = S\psi(k-1) + D(k)\hat{x}(k) + E_1(k) \tag{6.8}$$

where

$$S = \left[\begin{array}{ccc|ccc} O_{m\times m} & \cdots & O_{m\times m} & O_{m\times 1} & \cdots & O_{m\times 1} \\ & & O_{m\times m} & & & \\ & I_{m(p-1)} & \vdots & & O_{m(p-1)\times s} & \\ & & O_{m\times m} & & & \\ O_{1\times m} & \cdots & O_{1\times m} & 0 & \cdots & 0 \\ & & & & & 0 \\ & O_{(s-1)\times mp} & & & I_{s-1} & \vdots \\ & & & & & 0 \end{array}\right] \tag{6.9}$$

$$D(k) = \begin{bmatrix} H \\ 0 \\ \vdots \\ 0 \\ -L(\theta(k),k) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \; ; \; E_1(k) = \begin{bmatrix} v(k) \\ 0 \\ \vdots \\ 0 \\ r(k) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6.10}$$

and I_r, $O_{r\times s}$ denote an $r \times r$ identity matrix and an $r \times s$ zero matrix, respectively.

Define

$$z(k+1) = [\ \psi(k) \ \ \hat{x}(k+1)\ ]. \tag{6.11}$$

Inserting (5.23) into (5.1) and combining this result with (6.8) gives

$$z(k+1) = A(k)z(k) + E(k) \tag{6.12}$$

where

$$A(k) = \begin{bmatrix} S & D(k) \\ 0 & F(\theta(k),k) - G(\theta(k),k)L(\theta(k),k) \end{bmatrix} \tag{6.13}$$

$$E(k) = \begin{bmatrix} E_1(k) \\ M(k)v(k) + G(\theta(k),k)r(k) \end{bmatrix}. \tag{6.14}$$

By the theorem 5.2, $L(\theta(k),k)$ is bounded (thus $D(k)$ is bounded) and $(F(\theta(k),k) - G(\theta(k),k)L(\theta(k),k))$ is exponentially stable. Moreover, S is a constant exponentially stable matrix and thus there are $0 \leq \alpha < 1$ and $\beta \geq 1$ such that

$$\|\Phi(k+t,k)\| \leq \beta\alpha^t \ \ \forall k,t \in \mathcal{Z}^+. \tag{6.15}$$

where $\Phi(k+t,k)$ is the transition matrix of $A(k)$. Then from (6.12) we get

$$\|z(k+t)\| \le \beta\alpha^t\|z(k)\| + \sum_{i=0}^{t-1}\beta\alpha^{t-i-1}\|E(k+i)\|. \tag{6.16}$$

Assume that the reference signal $r(k)$ is bounded, i.e.,

$$|r(k)| \le R^* \ \forall k \in \mathcal{Z}^+ \ \text{ for some } R^*. \tag{6.17}$$

Since $G(\theta(k),k)$, $M(k)$ defined as (5.9) are uniformly bounded by Lemma 4.3 iii), $E(k)$ in (6.14) gives

$$\|E(k)\| \le (1+M^*)\|v(k)\| + (1+G^*)R^* \tag{6.18}$$

where

$$\begin{aligned} \|M(k)\| &\le M^* \\ \|G(\theta(k),k)\| &\le G^*. \end{aligned}$$

Now, it follows from (6.7) and (6.18) that

$$\begin{aligned} \|E(k)\| \ \le \ & (1+M^*)(KD+2\varepsilon)N^*\|\psi(k-1)\| + (1+G^*)R^* \\ & + (1+M^*)(KW+\varepsilon) \ \forall k \ge T. \end{aligned} \tag{6.19}$$

From (6.16) and (6.19), noting that (6.11) gives

$$\|\psi(k-1)\| \le \|z(k)\|$$

we get for any $k \ge T$

$$\|z(k+t)\| \le \beta\alpha^t\|z(k)\| + \sum_{i=0}^{t-1} c_1\alpha^{t-i-1}\|z(k+i)\| + \sum_{i=0}^{t-1} c_2\alpha^{t-i-1} \tag{6.20}$$

where

$$c_1 = \beta(1+M^*)(KD+2\varepsilon)N^*$$

$$c_2 = \beta(1+M^*)(KW+\varepsilon)+\beta(1+G^*)R^*.$$

Using the Gronwall Lemma which will be proved at the end of this chapter,

$$\begin{aligned}\|z(k+t)\|\alpha^{-t} \leq\ & (1+c_1/\alpha)^t\beta\|z(k)\| + c_2\alpha^{-t} + \\ & c_2\sum_{i=0}^{t-2}(1+c_1/\alpha)^{t-i-1}\alpha^{-i-1}\ \forall k \geq T.\end{aligned}$$

Thus, if $\alpha + c_1 < 1$, we get

$$\|z(k+t)\| \leq (\alpha+c_1)^t\beta\|z(k)\| + \frac{c_2}{1-(\alpha+c_1)}\ \forall k \geq T. \tag{6.21}$$

Choose D^* such that

$$D^* = \frac{1-\alpha}{\beta(1+M^*)KN^*} - \frac{2\varepsilon}{K} \tag{6.22}$$

which gives

$$\alpha + c_1 < 1\ \ \forall\, D < D^*.$$

Note that $N(k)$, $P(k)$, $M(k)$ and $G(\theta(k),k)$ are all uniformly bounded. Therefore, N^*, K, M^* and G^* exist which are not dependent on D. It then follows that $z(k+t)$ is bounded for any bounded reference signal, for any model uncertainties with $D_1 \leq D < D^*$, for any external disturbances with $W_1 \leq W$ and for any initial values of $z(k)$. ■

The essential idea in proving Theorem 6.1 is to use the equation (6.12) and the Gronwall lemma. The equation (6.12) was first introduced by Samson [8] and thereafter became an important tool to prove the closed-loop stability for indirect adaptive control systems. See also Ossman and Kamen [13].

We now prove the Gronwall lemma. For convenience in notation, the summation denoted by $\sum_{i=t_0}^{t_1}$ or the product denoted by $\prod_{i=t_0}^{t_1}$ will be taken as zero if $t_1 < t_0$.

Lemma 6.1 (Discrete-time version of Gronwall's lemma).

Suppose that $c(t)$, $d(t)$, $r(t)$ and $k(t)$ are nonnegative valued functions on $\mathcal{Z}^+$ such that

$$r(t) \leq c(t) + \sum_{i=0}^{t-1} k(i)r(i) \ \forall t \in \mathcal{Z}^+ \tag{6.23}$$

and

$$c(t) = \sum_{i=0}^{t-1} d(i) + c(0). \tag{6.24}$$

Then for any $t \in \mathcal{Z}^+$,

$$r(t) \leq \prod_{i=0}^{t-1}(1 + k(i))c(0) + d(t-1) + \sum_{i=0}^{t-2} \left\{ \prod_{j=i+1}^{t-1} (1 + k(j)) \right\} d(i). \tag{6.25}$$

Proof. The proof roughly follows the proof in the continuous-time case [21, 49]. Let

$$s(t) = c(t) + \sum_{i=0}^{t-1} k(i)r(i) \ \text{ for } t > 0 \ ; \ s(0) = c(0). \tag{6.26}$$

Then

$$r(t) \leq s(t). \tag{6.27}$$

Further

$$s(i) - s(i-1) = k(i-1)r(i-1) + d(i-1) \ \text{ for } i = 1, ..., t.$$

It then follows that

$$s(i) - (1 + k(i-1))s(i-1) \leq d(i-1) \ \text{ for } i = 1, ..., t.$$

Thus

$$s(t)-\prod_{i=0}^{t-1}(1+k(i))s(0)\leq d(t-1)+\sum_{i=0}^{t-2}\left\{\prod_{j=i+1}^{t-1}(1+k(j))\right\}d(i). \tag{6.28}$$

Since $s(0)=c(0)$ and $r(t)\leq s(t)$, we get (6.25). ∎

CHAPTER SEVEN
DESIGN EXAMPLE

In this chapter, we apply our adaptive control algorithm to a design of the normal acceration controller for missiles. For this study, we choose a generic air-to-air BTT (Bank-To-Turn) missile which has been used to develop an adaptive autopilot by Rockwell [50] and in a number of other studies (see, e.g., Caughlin and Bullock [51]). This model has an advantage that a large scale simulation based on the 6 DOF (Degree-Of-Freedom) equations of motion is available [52].

A typical trajectory is chosen for evaluation. This benchmark trajectory involves a hard turn which requires an immediate 90 degree bank and near maximum accerleration to intercept the target. For the initial range taken, the flight duration is just over 3 seconds. The guidance law is activated 0.4 second after launch, allowing the missile to clear the aircraft. The missile is accelerated for 2.6 seconds until the fuel is exhausted. The velocity changes from Mach 0.9 to Mach 2.6 and the dynamic pressure ranges from 900 psf to 8000 psf. Due to the rapid change of velocity and dynamic pressure, the plant parameters also change rapidly.

7.1 Plant Model

From the results of standard derivation for missile dynamics [53, 54], we have a linearized model for the longitudinal motion described as the second order state equation whose states are the velocity in z-direction $w(k)$ and the pitch rate $q(k)$. Assuming that the normal acceration $A_z(k)$ and the pitch rate $q(k)$ are directly

measurable, we have the following discrete-time plant model.

$$x(k) = F(k)x(k-1) + G(k)u(k-1) \tag{7.1}$$

$$y(k) = C(k)x(k) + D(k)u(k) \tag{7.2}$$

where

$$x'(k) = [w(k)\ q(k)]$$

$$y'(k) = [A_z(k)\ q(k)]$$

and $u(k)$ is the deflection angle of the control surface (the elevator or the carnard) and $F(k), G(k), C(k), D(k)$ are all time-varying matrices which have proper dimensions. $C(k)$ and $D(k)$ are in the form

$$C(k) = \begin{bmatrix} c(k) & 0 \\ 0 & 1 \end{bmatrix} \ ; \ D(k) = \begin{bmatrix} d(k) \\ 0 \end{bmatrix} \tag{7.3}$$

and $F(k)$ and $G(k)$ are general time-varying matrices. Figures 7.1 - 7.4 show entries of $F(k)$ and $G(k)$, $c(k)$ and $d(k)$ for the benchmark trajectory. This discrete-time model was obtained from the continuous-time model with the following sampling interval t_s (see, for details, [55]).

$$t_s = 0.01 \ \text{sec.}$$

Let $y^-(k)$ denote the output measured just before the control input $u(k)$ is applied. Since the discrete-time model is obtained from the continuous-time model by assuming that the control input is constant between two adjacent sampling intervals, $y^-(k)$ can be expressed as

$$y^-(k) = C(k)x(k) + D(k)u(k-1). \tag{7.4}$$

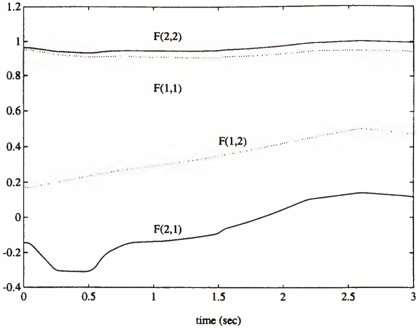

Figure 7.1: Entries of the system matrix $F(k)$. $F(i, j)$ denotes the (i, j)-th element of $F(k)$.

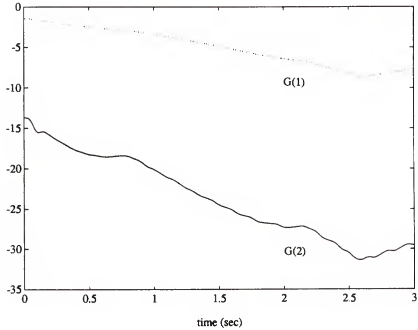

Figure 7.2: Entries of the input matrix $G(k)$. $G(i)$ denotes the i-th element of $G(k)$.

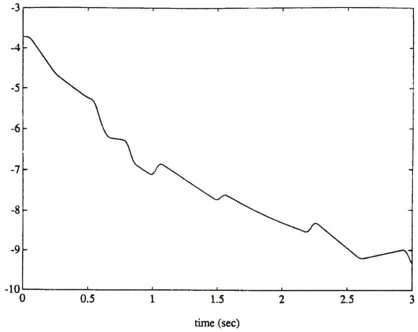

Figure 7.3: An element $c(k)$ of the output matrix $C(k)$.

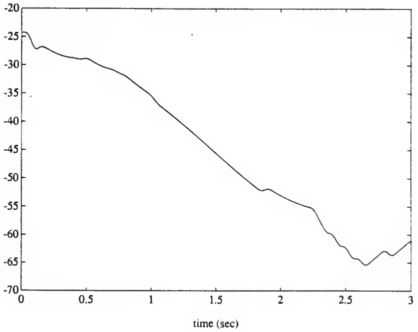

Figure 7.4: An element $d(k)$ of the matrix $D(k)$.

where $C(k)$ and $D(k)$ are as in (7.2).

We can choose either $y(k)$ or $y^-(k)$ for the parameter estimation. However, the choice will result in different control algorithms. If we choose $y(k)$, we will have an algebraic loop in the closed-loop system due to the direct term $D(k)u(k)$. To avoid this nuisance, we will choose $y^-(k)$. Then, we have to calculate the parameter estimates first, and the control input next.

Since $c(k) \neq 0$ (see Fig. 7.3), $\det\{C(k)\} \neq 0$. Therefore, using (7.1) we can restate $y^-(k)$ as

$$y^-(k) = A(k)y^-(k-1) + B_1(k)u(k-1) + B_2(k)u(k-2) \tag{7.5}$$

where

$$\begin{aligned} A(k) &= C(k)F(k)C^{-1}(k-1) \\ B_1(k) &= C(k)G(k) + D(k) \\ B_2(k) &= -A(k)D(k-1). \end{aligned} \tag{7.6}$$

The equation (7.5) is the plant model whose parameters are to be estimated.

It is known a priori that the dynamic pressure increases until the fuel is exhausted and thereafter it decreases. Moreover, it is known that the fuel is exhausted at

$$t_c = 2.6 \text{ seconds.}$$

As can be seen in Figures 7.1 - 7.4, the magnitudes of plant parameters increase almost linearly until the fuel is exhausted and thereafter they decrease. Therefore, the parameters can be approximated by linear curves until t_c seconds and after that by a monotonically decreasing function which goes to zero as time goes to infinity.

From this information, we will assume the structure of parameter variations as

$$f_1(k) = 1 \ ; \ f_2(k) = \begin{cases} t_s k & t_s k \leq t_c \\ t_c^2/(t_s k) & \text{otherwise} \end{cases} \tag{7.7}$$

where t_s denotes the sampling interval 0.01 sec. Thereby, we can write $A(k)$, $B_1(k)$ and $B_2(k)$ as

$$A(k) = \sum_{i=1}^{2}(A_i + \Delta A_i(k))f_i(k) \tag{7.8}$$

$$B_i(k) = \sum_{j=1}^{2}(B_{ij} + \Delta B_{ij}(k))f_j(k) \ \text{ for } i = 1,2. \tag{7.9}$$

Inserting (7.8) and (7.9) into (7.5) gives

$$y^-(k) = (\theta + \delta(k))'\phi(k-1) \tag{7.10}$$

where θ is the 8×2 parameter matrix, $\delta(k)$ is the 8×2 uncertainty matrix and $\phi(k)$ is the 8×1 regression vector, given by

$$\theta' = [\,A_1 \ A_2 \ B_{11} \ B_{12} \ B_{21} \ B_{22}\,] \tag{7.11}$$

$$\delta'(k) = [\,\Delta A_1(k) \ \Delta A_2(k) \ \Delta B_{11}(k) \ \Delta B_{12}(k) \ \Delta B_{21}(k) \ \Delta B_{22}(k)\,] \tag{7.12}$$

$$\phi'(k-1) = [\,f_1(k)y^-(k-1)' \ f_2(k)y^-(k-1)' \ f_1(k)u(k-1)$$
$$f_2(k)u(k-1) \ f_1(k)u(k-2) \ f_2(k)u(k-2)\,]. \tag{7.13}$$

7.2 Estimated Model

Associated with (7.10), the prediction error $e(k)$ is given by

$$e(k) = y^-(k) - \theta'(k-1)\phi(k-1) \tag{7.14}$$

Table 7.1: Possible parameter ranges

Maximum values	$\begin{bmatrix} 1.05 & -0.3 & 0.1 & 2.5 & 10 & 25 & 50 & 25 \\ 0.2 & 1.05 & 0.1 & 0.1 & -5 & 25 & 2 & 25 \end{bmatrix}$
Minimum values	$\begin{bmatrix} 0.85 & -6 & -0.1 & -2.5 & -50 & -25 & -50 & -25 \\ 0.04 & 0.85 & -0.1 & -0.1 & -50 & -25 & -2 & -25 \end{bmatrix}$

where $\theta(k-1)$ is the estimate of θ at time $(k-1)$.

Using the linearized model for the benchmark trajectory (Fig. 7.1 - 7.4), the nominal $A(k)$, $B_1(k)$ and $B_2(k)$ were calculated. Based on these curves, the possible ranges of the tuned parameters were taken as in Table 7.1.

In order to use the state space model (7.1) and (7.4), we must find $F(k)$, $G(k)$, $C(k)$ and $D(k)$ from the estimates of $A(k)$, $B_1(k)$ and $B_2(k)$ using (7.6). However, equation (7.6) cannot be solved uniquely for $F(k)$, $G(k)$, $C(k)$ and $D(k)$. Therefore, we use a different model which is easy to construct from the estimated $A(k), B_1(k)$ and $B_2(k)$.

Let

$$z(k) = C(k)x(k). \tag{7.15}$$

Then, it follows from the original state equations (7.1),(7.4) and the definition of $y^-(k)$ (7.5) that

$$z(k) = y^-(k) - D(k)u(k-1) \tag{7.16}$$

$$z(k+1) = A(k+1)z(k) + B(k+1)u(k) \tag{7.17}$$

where

$$D(k) = -A^{-1}(k+1)B_2(k+1) \tag{7.18}$$

$$B(k+1) = B_1(k+1) - D(k+1). \tag{7.19}$$

Remark 7.1. According to (3.1), this example corresponds to the case that $p = 1$ and $s = 2$. Thus, we have an observable realization of (7.10) in the form of (3.20) and (3.21). This result is exactly same as (7.16)-(7.19).

The set composed of the possible parameter ranges (Table 7.1) contains some points which do not satisfy the reachability condition or the invertibility condition of $A_p(\theta, k)$ (Assumption 6 or 7 in Section 3.1). If we exclude such points from this set (call the resulting set Ω^*), Ω^* will not be a connected set and thus not a hypercube as in Assumption 6. Furthermore, any hypercube as a subset of Ω^* which satisfies the reachability condition and the invertibility condition of $A_p(\theta, k)$ (Assumption 6 and 7 in Section 3.1) is so small in size that we do not need to use adaptive control.

An alternative to avoid these technical difficulties is to weaken Assumption 6 and 7 in Section 3.1 as the following which allow a hypercube Ω with reasonable size.

A1. There is $\varepsilon > 0$ such that for any $\theta \in \Omega$, an observable realization of

$$y(k) = \theta' \phi(k-1)$$

satisfies

$$\liminf_k \{|\det(Y_N(\theta, k))|\} \geq \varepsilon$$

where $Y_N(\theta, k)$ is its reachability grammian.

A2. If $p < s$, then there is $\varepsilon_1 > 0$ such that for any $\theta \in \Omega$,

$$\liminf_k \{|\det(A_p(\theta, k))|\} \geq \varepsilon_1$$

where

$$A_p(\theta, k) = \sum_{j=1}^{r} A_{pj}\, f_j(k)$$

and $A_{pj},\ j = 1, \ldots, r$ are submatrices of θ' for any $\theta \in \Omega$.

Remark 7.2. Assumption A1 implies that for a fixed $\theta \in \Omega$, $\det(Y_N(\theta, k))$ can be zero, but it must not occur infinitely many times. In other words, there exists a positive integer T such that for any $\theta \in \Omega$ its observable realization $(F(\theta, k), G(\theta, k))$ is uniformly reachable in N steps for $k \in [T, \infty)$. When the estimated model is not reachable at time $k \leq T$, we use any bounded feedback gains.

As for the example in this chapter, this alternative is not very helpful because the flight duration is just 3 seconds. However, we will show that the set of the possible parameter ranges (call it Ω) satisfies assumptions A1 and A2 and use this set in the computer simulations to see how the projection scheme works.

Now we check that the set Ω satisfies assumptions A1 and A2. For the system (7.17), the 2-step reachability matrix $R_2(\theta, k)$ for $\theta \in \Omega$ is given by

$$R_2(\theta, k) = [\, B(\theta, k+1) \;\; A(\theta, k+1)B(\theta, k)\,]. \tag{7.20}$$

Since $f_2(k) \to 0$ as $k \to \infty$, $R_2(\theta, k)$ converges to a certain matrix $R_2(\theta, \infty)$ where

$$R_2(\theta, \infty) = A_1^{-1}[\, A_1 B_{11} + B_{21} \;\; A_1(A_1 B_{11} + B_{21})\,] \tag{7.21}$$

and A_1, B_{11}, B_{21} are submatrices of θ as in (7.11). From the data in Table 7.1, it can be easily shown that A_1 is invertible for any $\theta \in \Omega$. To show that $\det\{R_2(\theta, \infty)\} \neq 0$, let

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = A_1 B_{11} + B_{21}.$$

Then, from Table 7.1, we get $e_2 < 0$. Define

$$t = e_1/e_2.$$

Then, $\det\{R_2(\theta,\infty)\} = 0$ implies that

$$a_{21}t^2 + (a_{22} - a_{11})t - a_{12} = 0 \tag{7.22}$$

where

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

In order for the equation (7.22) to have real solutions, it should satisfy

$$(a_{22} - a_{11})^2 + 4a_{21}a_{12} \geq 0 \tag{7.23}$$

which, from the possible ranges of a_{11} and a_{22} in Table 7.1, gives

$$a_{21}a_{12} \geq -0.01. \tag{7.24}$$

But, we have from the possible ranges of a_{11} and a_{12}

$$a_{21}a_{12} \leq -0.012$$

which contradicts (7.24). Therefore, for any $\theta \in \Omega$ $\det\{R_2(\theta,\infty)\} \neq 0$ which implies that Ω satisfies Assumption 6 in Section 3.1. However, as discussed in Section 5.5, the augmented estimated model based on the internal model principle may not satisfy the reachability assumption like A1 with this Ω.

7.3 Controller Structure

For $A_z(k)$ to track a commanded acceleration $r(k)$ so that the track error

$$|A_z(k) - r(k)|$$

is suitably small, we use the internal model principle. Suppose that $r(k)$ is a step command, i.e.,

$$r(k) = a \text{ for some } a \in \mathcal{R}.$$

By the standard procedure of augmenting the system model, we define

$$v(k+1) = v(k) + (A_z(k) - r(k)) \tag{7.25}$$

Since $A_z(k)$ is the first component of $y(k)$, we have from (7.2) and (7.15)

$$v(k+1) = v(k) + [1 \quad 0](z(k) + D(k)u(k)) - r(k). \tag{7.26}$$

Let $\hat{z}(k)$ and $\hat{v}(k)$ denote the estimated $z(k)$ and $v(k)$, respectively. Then, from (7.17) and (7.26), using the latest estimates, we have the following augmented estimated model.

$$\begin{bmatrix} \hat{z}(k+1) \\ \hat{v}(k+1) \end{bmatrix} = \begin{bmatrix} A(\theta(k), k+1) & \begin{matrix} 0 \\ 0 \end{matrix} \\ 1 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{z}(k) \\ \hat{v}(k) \end{bmatrix} +$$

$$\begin{bmatrix} B(\theta(k), k+1) \\ D_1(\theta(k), k) \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 0 \\ -r(k) \end{bmatrix} \tag{7.27}$$

where

$$D_1(\theta(k), k) = [1\,0]D(\theta(k), k).$$

For the system (7.27), Kleinman's method gives the following control law.

$$u(k) = -L(\theta(k), k) \begin{bmatrix} \hat{z}(k) \\ \hat{v}(k) \end{bmatrix}. \tag{7.28}$$

As mentioned in Section 5.5 the rachability of the estimated model does not imply the rechability of the augmented estimated model, so the control law (7.28) may result in the instability of the closed-loop. To avoid this, we take the second approach suggested in Section 5.5. If the augmented model converges to a system which is not reachable, the control law (7.28) will switch to that of the estimated model without augmentation.

In this example, the number of outputs is the same as the number of states, because $p = 1$. Thus, no dynamic observer is required. The non-dynamic state observer is obtained from (7.16) as

$$\hat{z}(k) = y^{-}(k) - D(\theta(k), k)u(k-1). \tag{7.29}$$

On the other hand, since inserting (7.16) into (7.26) gives

$$\begin{aligned} v(k) &= v(k-1) - r(k-1) + \\ &\quad [1\ 0]\{y^{-}(k-1) + D(k-1)(u(k-1) - u(k-2))\}. \end{aligned} \tag{7.30}$$

$\hat{v}(k)$ is then given by

$$\begin{aligned} \hat{v}(k) &= \hat{v}(k-1) - r(k-1) + \\ &\quad [1\ 0]\{y^{-}(k-1) + D(\theta(k), k-1)(u(k-1) - u(k-2))\}. \end{aligned} \tag{7.31}$$

7.4 Simulation Results

To see the performance of the proposed adaptive control algorithm implemented in the normal acceleration controller, computer simulations were generated with different design parameters and initial values, by using the linearized aerodynamic

model or the large scale simulation program. Among these results, we demonstrate the following four cases obtained from the large scale simulations. For all 4 cases, the update intervals of control and estimation are chosen to be 0.01 sec.

Case 1.

This case shows how the proposed algorithm works when the estimation scheme takes the structure of the parameter variations as described in (7.7). We do not use any dead zone in this case. Also, no projection scheme is used. To ensure that the missile clears the aircraft from which it was discharged the guidance law is not activated for 0.4 seconds after launch, however in this interval the pitch rate feedback loop is active regulating the pitch rate to zero. For this purpose, the following constant feedback gain L_0, with which the closed-loop of the nominal plant (Fig. 7.1 - 7.4) is stable, will be used:

$$L_0 = [\,0\ \ -0.13\ 0\,]. \tag{7.32}$$

The initial value of the parameter esimate $\theta(k)$ was chosen within the set Ω and the initial value of the covariance matrix $P(k)$ was taken as a diagonal matrix whose diagonal elements approximate the squares of possible parameter ranges.

$$\theta(0) = \begin{bmatrix} 0.95 & -0.5 & 0 & -2 & -20 & 10 & 25 & 10 \\ 0.05 & 0.95 & -0.05 & 0 & -12 & -10 & 1 & -1 \end{bmatrix} \tag{7.33}$$

$$P(0) = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix} \qquad (7.34)$$

A guide to choose the value of a design parameter $N(0)$ which is used to calculate a forgetting factor $\lambda(k)$ is to follow the constant information principle, i.e.,

$$N(0) = \tilde{\theta}'(0)P^{-1}(0)\tilde{\theta}(0).$$

with the guess of $\tilde{\theta}(0)$. If we choose a larger $N(0)$, the resulting $\lambda(k)$ will be closer to 1. In this example, $N(0)$ was chosen as

$$N(0) = 1.$$

Since no dead zone is used,

$$D = 0.$$

Figures 7.5 - 7.8 show a set of results obtained from this simulation. In Fig. 7.7 - 7.8, the nominal system parameters (which were obtained from Fig. 7.1 - 7.4) are marked "Reference."

Case 2.

This case shows how the proposed algorithm works when the estimation scheme takes the parameters to be estimated as piecewise constant, i.e.,

$$f_1(k) = 1.$$

As in Case 1, neither projection nor dead zone is used. The initial value $\theta(0)$ was chosen so that the elements of $A(k)$, $B_1(k)$ and $B_2(k)$ have the same values as in Case 1 at 0.4 second and $P(0)$ was chosen similar to Case 1. The value of $N(0)$ was selected so that the tracking errors of the normal acceleration became as small as possible.

$$\theta(0) = \begin{bmatrix} 0.95 & -1.3 & -16 & 29 \\ 0.03 & 0.95 & -16 & 0.6 \end{bmatrix} \tag{7.35}$$

$$P(0) = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 1000 \end{bmatrix} \tag{7.36}$$

$$N(0) = 0.0001. \tag{7.37}$$

The results are shown in Fig. 7.9 - 7.12. As expected, the simulation results showed poor performance. Further disscussion will be given in Section 7.5.

Case 3.

This case agrees with Case 1 except for $\theta(0)$ and $N(0)$. The initial value $\theta(0)$ which is more inaccurate than in Case 1 with respect to the reference was taken. The value of $N(0)$ was selected so that the tracking errors of the normal acceleration became as small as possible as in Case 2.

$$\theta(0) = \begin{bmatrix} 0.8 & -0.5 & 0 & 0 & 10 & 0 & 10 & 0 \\ 0 & 1 & 0 & 0 & -10 & 0 & 0 & 0 \end{bmatrix}. \tag{7.38}$$

$$N(0) = 0.01. \tag{7.39}$$

A set of responses for this simulation is shown in Fig. 7.13 - 7.16.

Case 4.

This case agrees with Case 2 except for the value of $N(0)$ which is much larger than in Case 2.

$$N(0) = 1. \tag{7.40}$$

See Fig. 7.17 - 7.20. The resulting forgetting factors were much closer to 1 than Case 2. Consequently, the parameter estimates converged to some constants when the system was stable, which caused bad oscillations at 1.3 second and 2.6 second.

7.5 Discussion

Since we use a variable forgetting factor, the estimation algorithm is allowed, to some extent, to track the time-varying parameters. However, when we compare the results of Case 1 or Case 3 (where parameters are assumed as linear curves) with those of Case 2 or Case 4 (where parameters are asumed as piecewise constants), we see that the performance of the latter is much worse than the former. This is mainly due to the following facts.

First, when the value of $N(0)$ used was relatively large (Case 4), while the system was stable the forgetting factors were close to 1 and consequently the elements of $F(\theta(k), k)$ and $G(\theta(k), k)$ converged to some constants (See Fig. 7.19). However, in Case 1 or Case 3, these elements vary due to $f_2(k)$, even if the estimated parameters are constant. It is obvious that the assumed structure of parameter variations in Case 1 or Case 3 has the ability of more accurate time-varying elements representation so that the resulting control law could give better performance if the parameters are properly identified. On the other hand, when the value of $N(0)$ used was relatively small (Case 2), the ability of tracking time-varying parameters

was improved, but the resulting estimated parameters changed rapidly which caused oscillatory motions (see Fig. 9).

Secondly, when we calculate the feedback gains using Kleinman's method, future values of the system and input matrices are required. This implies that the time-varying feedback gains depend on the structure of parameter variations (described as known functions $f_i(k), i = 1, ..., r$). In Case 2 or Case 4, the assumed parameter structure

$$f_1(k) = 1$$

takes the parameters as constant. Therefore, the resulting control law becomes a controller based on the frozen coefficients which may fail to work for the plants which are not slowly time-varying (see Section 2.7).

In the theory developed in Chapter Three to Chapter Six, the set Ω had a number of assumed properties and projection to Ω was used whenever the parameter estimates were outside of Ω. In fact, in the simulations no significant difference of performance with and without projection could be seen. See Fig. 7.21 - 7.23. The assumed structure of parameter variations, $\theta(0)$ and $P(0)$ used in these simulations are same as in Case 3 and no dead zone was used. However, $N(0)$ was taken to be 1 which is much larger than in Case 3. Without projection, Fig. 7.21 shows the Frobenius norm of the distance from $\theta(k)$ to Ω which is denoted by $\|\theta(k) - \Omega\|_F$ (see, for the definition of the Frobenius norm, Section 5.3 or [47]) and Figures 7.22 - 7.23 show the response and the feedback gains with and without projection. In this example, the tracking of the commanded acceleration is good in both cases except for a switching transient at 1.7 second. We may conclude, for this example, that the projection scheme is unnecessary complication in practice.

For each of 4 cases the norm of feedback gains was properly bounded (see Fig. 7.8, 7.12, 7.16 and 7.20). This implies that for each case the augmented estimated model did not violate the reachability condition (Assumption 6.a in Section 5.5).

D is a design parameter related with the norm of the plant uncertainties. To prove the robust stability of the closed loop, we assumed that $D \geq D_1$ where D_1 is the supremum of the plant uncertainties. However, in practical applications the value of D should be as small as possible because, in general, the robustness can be achieved at the price of performance. Using a dead zone shows slightly better performance than otherwise, when a design parameter D is properly selected. See Fig. 7.24 - 7.26 where the results marked "Without Dead Zone" represent the case "Without Projection" in Fig. 7.21 - 7.23 while the results marked "With Dead Zone" represent the same case except that $D = 0.01$. As shown in Fig. 7.24 -7.26, the difference is negligible. In spite of various plant uncertainties, the large scale simuations without deadzone gave good performance. This fact implies that our adaptive control algorithm without dead zone is, to a good extent, robust to the plant uncertainties.

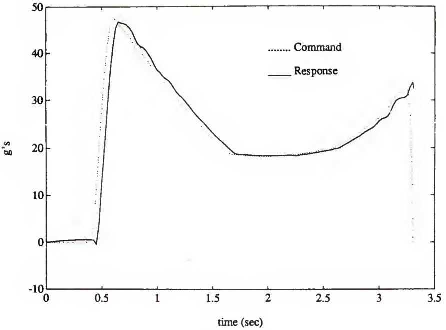

Figure 7.5: Response of the normal acceleration N_z for Case 1.

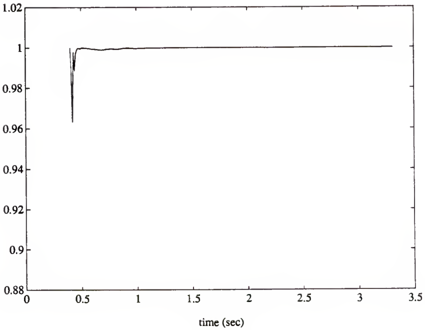

Figure 7.6: $\lambda(k)$ for Case 1.

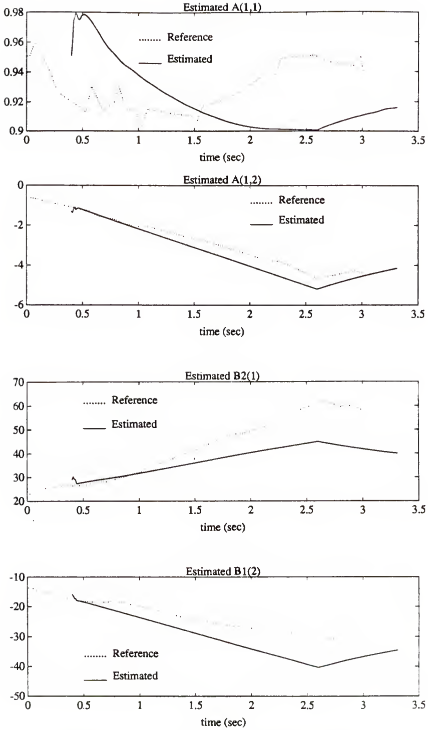

Figure 7.7: Some elements of estimated $A(k)$, $B_1(k)$ and $B_2(k)$ for Case 1. $A(i,j)$ denotes the (i,j)-th element of $A(k)$ and $Bi(j)$ denotes the j-th element of $B_i(k)$.

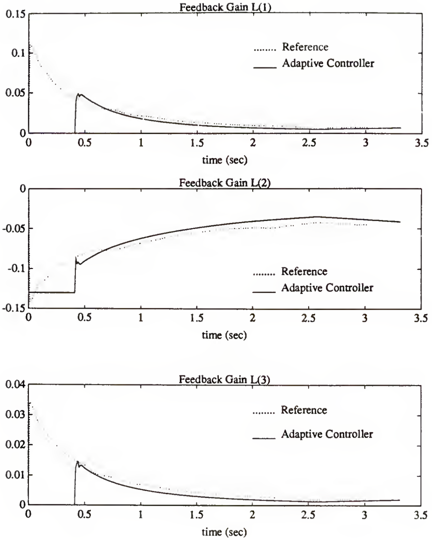

Figure 7.8: Feedback gain $L(k)$ for Case 1. $L(j)$ denotes the j-th element of $L(k)$.

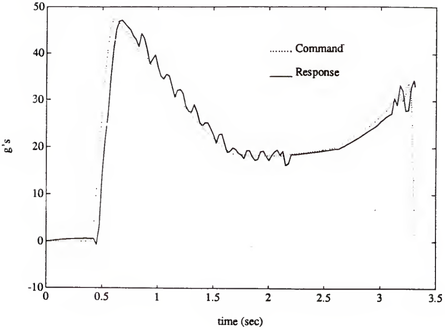

Figure 7.9: Response of the normal acceleration N_z for Case 2.

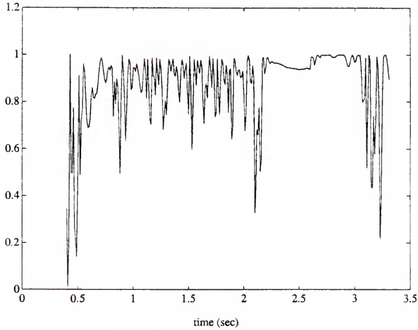

Figure 7.10: $\lambda(k)$ for Case 2.

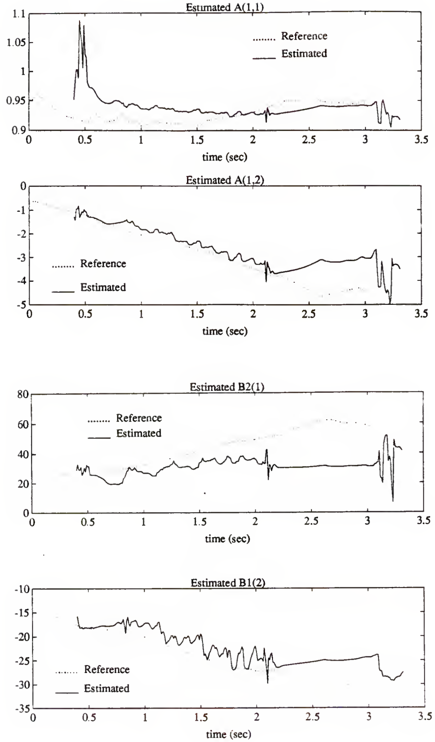

Figure 7.11: Some elements of estimated $A(k)$, $B_1(k)$ and $B_2(k)$ for Case 2. $A(i,j)$ denotes the (i,j)-th element of $A(k)$ and $Bi(j)$ denotes the j-th element of $B_i(k)$.

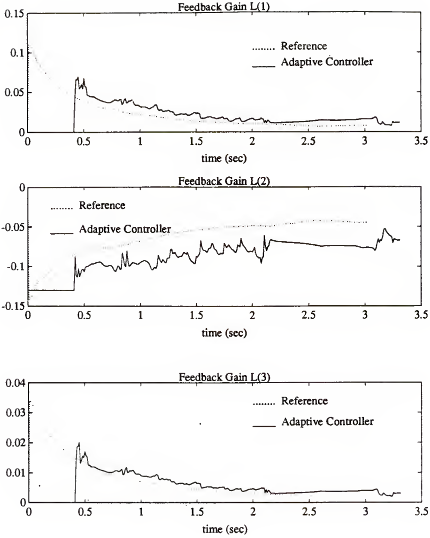

Figure 7.12: Feedback gain $L(k)$ for Case 2. $L(j)$ denotes the j-th element of $L(k)$.

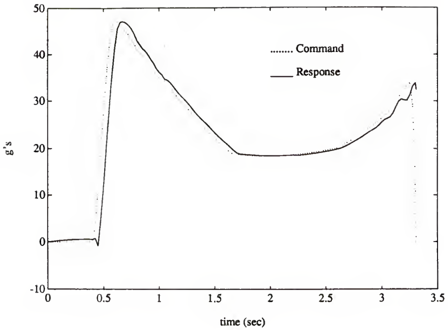

Figure 7.13: Response of the normal acceleration N_z for Case 3.

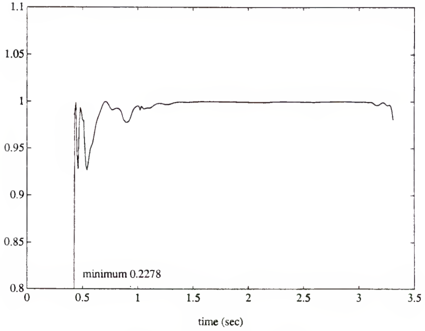

Figure 7.14: $\lambda(k)$ for Case 3.

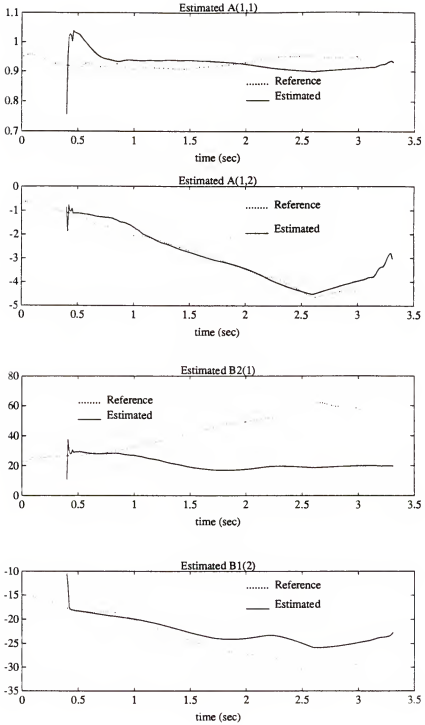

Figure 7.15: Some elements of estimated $A(k)$, $B_1(k)$ and $B_2(k)$ for Case 3. $A(i,j)$ denotes the (i,j)-th element of $A(k)$ and $Bi(j)$ denotes the j-th element of $B_i(k)$.

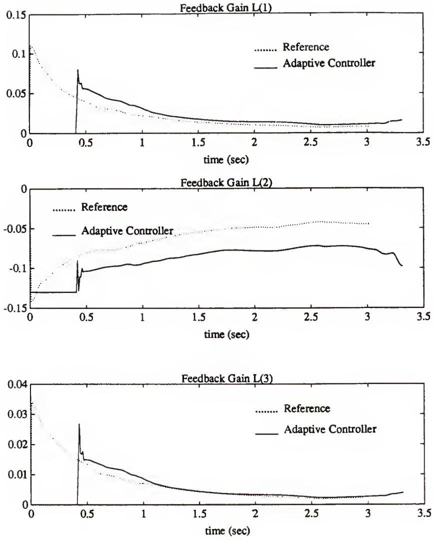

Figure 7.16: Feedback gain $L(k)$ for Case 3. $L(j)$ denotes the j-th element of $L(k)$.

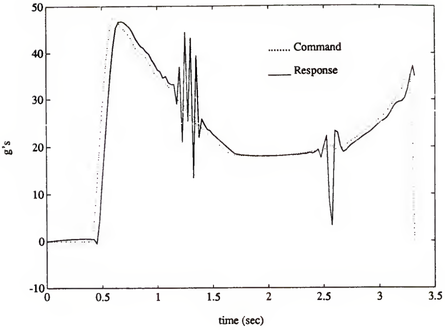

Figure 7.17: Response of the normal acceleration N_z for Case 4.

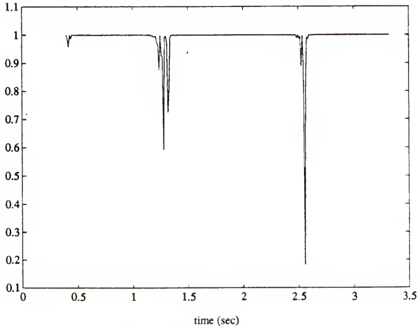

Figure 7.18: $\lambda(k)$ for Case 4.

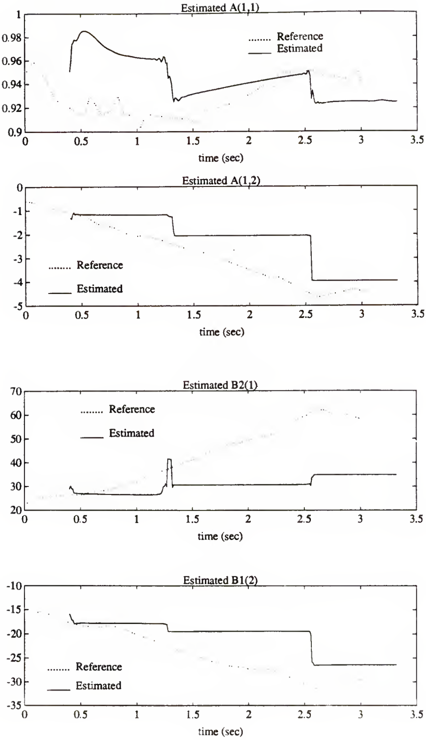

Figure 7.19: Some elements of estimated $A(k)$, $B_1(k)$ and $B_2(k)$ for Case 4. $A(i,j)$ denotes the (i,j)-th element of $A(k)$ and $Bi(j)$ denotes the j-th element of $B_i(k)$.

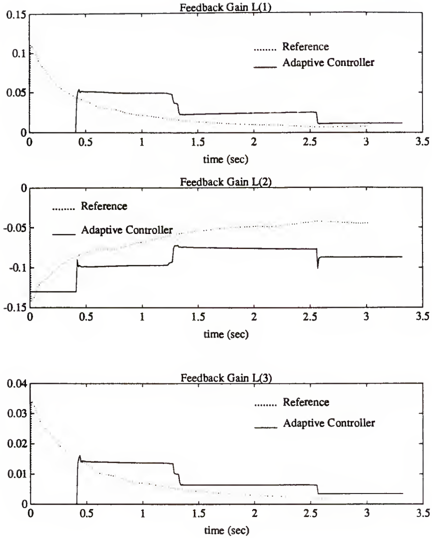

Figure 7.20: Feedback gain $L(k)$ for Case 4. $L(j)$ denotes the j-th element of $L(k)$.

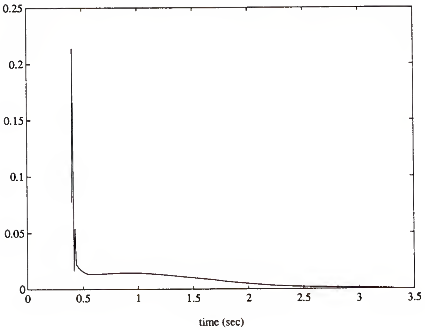

Figure 7.21: $\|\theta(k) - \Omega\|_F$ for the case without projection.

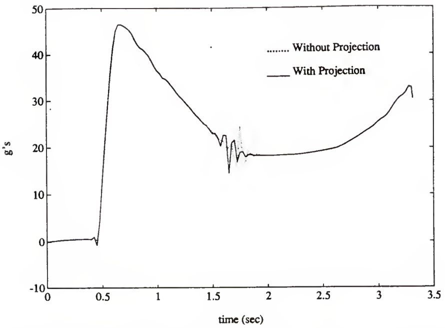

Figure 7.22: Responses of the normal acceleration N_z with and without projection.

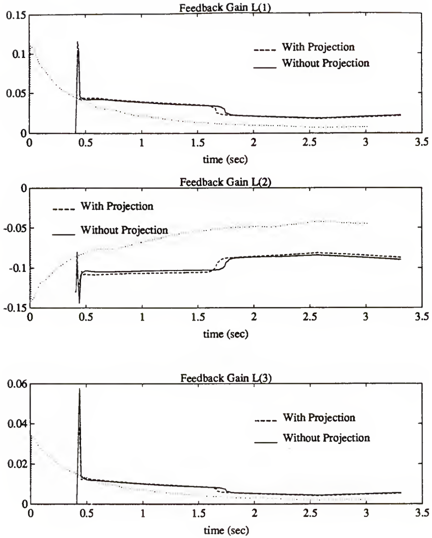

Figure 7.23: Feedback gains $L(k)$ with and without projection. $L(j)$ denotes the j-th element of $L(k)$.

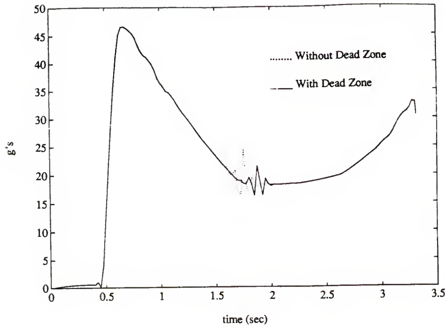

Figure 7.24: Responses of the normal acceleration N_z with dead zone ($D = 0.01$) and without dead zone.

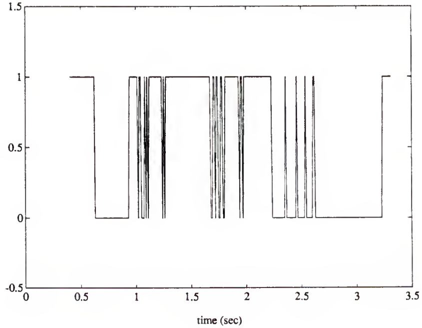

Figure 7.25: $\alpha(k)$ with $D = 0.01$.

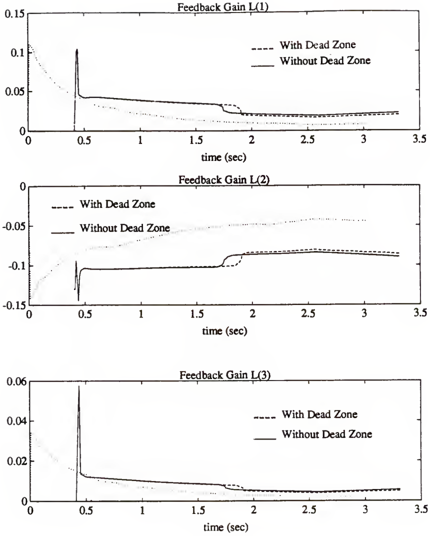

Figure 7.26: Feedback gains $L(k)$ with dead zone ($D = 0.01$) and without dead zone. $L(j)$ denotes the j-th element of $L(k)$.

CHAPTER EIGHT
CONCLUSIONS

8.1 Summary

In this dissertation, an indirect adaptive control algorithm for linear systems was presented. This algorithm can be applied to linear time-invariant systems as well as to linear time-varying systems with rapidly varying parameters. A key feature of the proposed algorithm is the use of an assumed structure of parameter variations which enables it to cover a wide class of linear systems.

The proposed approach consists of two parts, an estimation algorithm and a control algorithm. The estimation algorithm is based on the recursive least squares algorithm but includes a forgetting factor, a dead zone and a projection scheme which keeps the estimates in a compact set Ω. It was shown that the robustness of the closed-loop system can be achieved by using a dead zone in the estimation algorithm. The robustness is defined in a sense that if the actual plant is almost like the tuned model (i.e., the plant uncertainties are small enough) then the closed-loop system will be stable. A forgetting factor based on the constant information principle is introduced in the estimation algorithm. The constant information principle provides a forgetting factor with properties which are very useful to prove the closed-loop stability. A key property is that the infinite product of forgetting factors is uniformly bounded below by some positive constant (see Lemma 4.2). It then follows from this property that the covariance matrix $P(k)$ and the estimates $\theta(k)$

are uniformly bounded and converge. As a control algorithm, Kleinman's method is used. Its advantage over other methods is ease of implementation, especially when the reachability index N is greater than the system order (which can occur when the system is time-varying).

To prove the closed-loop stability, a bounded sequence of feedback gains which makes the estimated model exponentially stable and an estimation algorithm in which $\|\theta(k) - \theta(k-1)\|$ converges to zero and Lemma 4.3 ii) is satisfied were needed. Therefore, Kleinman's method can be replaced by any other controller and the proposed estimation algorithm by any other algorithm if they satisfy the conditions stated above. The validity of a controller based on the internal model principle (an example is shown in Section 5) can be claimed from such a point of view.

We applied this adaptive control algorithm to the design of the normal acceleration controller for a typical missile model. With computer simulations, it has been found that the proposed algorithm gives excellent performance. We arrive at the following conclusions:

- The use of the structure of parameter variations is a key to solve the adaptive control problems for linear time-varying systems with rapidly varying parameters.

- Although the theory developed required projection, in practice in the simulations projection was neither beneficial nor required.

- Even though the theory developed required assumptions on the bounds to plant uncertainties and use of dead zone, the large scale simulations without any dead zone gave good performance, which implies that the use of dead zone may not be required in practice.

In the proposed algorithm, many techniques are combined. We use a forgetting factor, a dead zone, signal normalization and a projection scheme for the estimation algorithm. We also use an adaptive observer, Kleinman's control law and the internal model principle (if necessary) for the control algorithm. For this combination, we showed the uniform boundedness of the covariance matrix $P(k)$ and the estimate $\theta(k)$. We further showed the convergence properties of $P(k)$ and $\|\theta(k) - \theta(k-1)\|$, and the robust stability for linear time-varying systems which are not necessarily slowly time-varying. Therefore, the proposed algorithm not only allows design flexibility but also covers a wide class of linear systems.

8.2 Problems and Future Research Work

Research work tends to uncover new problems as the work progresses. The work in this dissertation follows this rule. In this section a few problems for future research encountered in this investigation are mentioned. In this dissertation, indirect adaptive control of time-varying systems which are not slowly time-varying has been considered. If it is possible to get some information about the structure of the time-variation for the control parameters instead of system parameters, then we will be able to apply the idea of structured parameter variations to the direct adaptive control scheme for linear time-varying systems.

The work presented in this dissertation could be strengthened by weakening some of the restrictive assumptions. Assumption 6 in Section 3.1 required the existence and knowledge of a hypercube Ω as a subset of Π, the set of all parameter vectors satisfying the reachability condition for the estimated models, which is in practice difficult to determine. Since ignoring the required projection lead to no difficulties in the simulations it might be possible to show that an improved param-

eter vector inside of Π could be found when a parameter estimate was outside of Π without requiring total knowledge of the set Π. Another restrictive assumption was Assumption 6.a in Section 5.5 which requires a hypercube as a subset of Γ where Γ is the set of all parameter vectors satisfying the reachability condition for the augmented estimated model. Again, it might be possible to find a better substitute in Γ when a parameter estimate was outside of Γ without requiring total knowledge of the set Γ. We leave it as future research work to weaken these restrictive assumptions.

It is well known that if the plant is properly excited by external signals so that the plant input is persistently exciting, complete parameter identification is possible. To excite the plant externally, dither or probing signals are used in practice. These signals were not considered in this dissertation. Such external signals are sometimes undesirable from a control point of view. Moreover, even for the linear time-invariant plants guaranteeing persistent excitation in the presence of unmodelled plant uncertainties is still an open problem.

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BIOGRAPHICAL SKETCH

Chanho Song was born in Eui-Jung-Bu, Korea, on August 30, 1953. He received the B.S. and the M.S. degrees in electrical engineering from Seoul National University, Korea, in 1975 and 1977, respectively. From 1977 to 1984, he was with the Agency for Defence Development, Korea. During this period, he worked as a research engineer in the Division of Guidance and Control. Since January 1985, he has been working toward the Ph.D. degree at the University of Florida.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

T. E. Bullock

T. E. Bullock, Chairman
Professor of Electrical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

J. Hammer

J. Hammer
Associate Professor of Electrical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

H. A. Latchman
Assistant Professor of Electrical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

S. A. Svorony

S. A. Svoronos
Associate Professor of Chemical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

U. H. Kurzweg

U. H. Kurzweg
Professor of Aerospace Engineering,
Mechanics, and Engineering Science

This dissertation was submitted to the Graduate Faculty of the College of Engineering and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 1989

Hubert A. Bevis

Dean, College of Engineering

Dean, Graduate School